Joint Program Exam in Real Analysis

September 2021

Instructions:

1. Print your student ID and the problem number on each page. Write on one side of each paper sheet only. Start each problem on a new sheet. Write legibly using a dark pencil or pen.

2. You may use up to three and a half hours to complete this exam.

3. The exam consists of 8 problems. All the problems are weighted equally. **You need to do 7 of the 8 problems for full credit.**

4. For each problem which you attempt try to give a complete solution. Completeness is important: a correct and complete solution to one problem will gain more credit than two half solutions to two problems. Justify the steps in your solutions by referring to theorems by name, when appropriate, and by verifying the hypotheses of these theorems. You do not need to reprove the theorems you used.

5. Throughout this exam m and m_n denote Lebesgue measure on $\mathbb{R}$ and $\mathbb{R}^n$, respectively, or on one of their subsets. The words 'measurable' and 'integrable' refer to Lebesgue measure and all integrals are Lebesgue integrals. Integration with respect to m is also denoted by dx (or similar). $L^p(E)$ denotes the L^p-space with respect to Lebesgue measure on the Lebesgue measurable set E, with corresponding norm $\|\cdot\|_p$.

1. For each of the following statements decide whether it is true or false. If true, give a proof; if false, give a counter-example.

   a) Let $\mathbb{Q}$ be the set of rational numbers in $(0, 1)$ and $\varepsilon > 0$. There is an open set V containing $\mathbb{Q}$ such that $m(V) < \varepsilon$.

   b) There is a closed set $E \subset [0, 1]$ with a positive Lebesgue measure and with empty interior.

2. For each of the following statements decide whether it is true or false. If true, give a proof; if false, give a counter-example.

   (a) $L^1(\mathbb{R}) \cap L^3(\mathbb{R}) \subseteq L^2(\mathbb{R})$.

   (b) $L^2(\mathbb{R}) \subseteq L^1(\mathbb{R}) \cup L^3(\mathbb{R})$.

3. Let $E \subset \mathbb{R}^d$ be closed and bounded. Assume that $\{f_n\}_{n=1}^{\infty}$ is a Cauchy sequence in $L^2(E)$ and $K(x, y)$ is continuous on $E \times E$. Let

$$F_n(x) = \int_E K(x, y) f_n(y)\, dy.$$

   Prove that F_n converges to some function F in $L^\infty(E)$.

4. Let $\{f_n\}$ be a sequence in $L^{4/3}([0, 1])$ such that $f_n \to 0$ in measure as $n \to \infty$ and $\int_0^1 |f_n|^{4/3}\, dx \leq 1$. Prove that $\int_0^1 |f_n|\, dx \to 0$ as $n \to \infty$.

5. Show that the function f is Lipschitz continuous on $[0, 1]$, namely, $|f(x) - f(y)| \leq M|x - y|$ for all $x, y \in [0, 1]$ and some constant $M > 0$ if and only if f is absolutely continuous on [0,1] and $|f'| \leq M$ a.e. in $[0, 1]$.

6. Let $f : \mathbb{R} \to \mathbb{R}$ be Lebegue measurable and $f \in L^1(\mathbb{R})$. Prove that

$$\varphi(x) = \int_{\mathbb{R}} \frac{f(x - y)}{y^2 + 1} dy$$

   exists for almost all $x \in \mathbb{R}$, $\varphi(x) \in L^1(\mathbb{R})$ and $\|\varphi\|_1 \leq \pi \|f\|_1$. Make sure to justify all the steps.

7. Let $E \subset [0, 1]$ be a measurable set with $m(E) > 0$ and $n \in \mathbb{N}$. Prove that there exist n disjoint measurable sets E_i, $1 \leq i \leq n$, such that

$$E = \bigcup_{i=1}^{n} E_i \quad \text{and} \quad m(E_i) = m(E)/n.$$

8. Prove that the following limit exists and determine its value:

$$\lim_{n\to\infty} \int_0^\infty ne^{-nx} \sin(1/x)\, dx.$$