Joint Program Exam in Real Analysis

September 10, 2019

Instructions:

1. Print your student ID (but not your name) and the problem number on each page. Write on one side of each paper sheet only. Start each problem on a new sheet. Write legibly using a dark pencil or pen.

2. You may use up to three and a half hours to complete this exam.

3. The exam consists of 8 problems. All problems are weighted equally.

4. For each problem which you attempt try to give a complete solution and justify carefully your reasoning. Completeness is important: a correct and complete solution to one problem will gain more credit than two half solutions to other problems. Justify the steps in your solutions by referring to theorems by name, when appropriate, and by verifying the hypotheses of these theorems. You do not need to reprove the theorems you used.

5. $\mathbb{R}$ denotes the set of real numbers, $\mathbb{N}$ denotes the set of positive integers, $\mathrm{m}(A)$ refers to the Lebesgue measure of the set $A \subset \mathbb{R}^d$, "measurable" refers to Lebesgue measurable, and "a.e." means almost everywhere with respect to Lebesgue measure unless noted otherwise. Instead of $d\mathrm{m}$ we sometimes write dx, dt, etc. referring to the variable to be integrated. $L^p(X, \mu)$ denotes the Lebesgue space of order p with respect to the positive measure μ and $\| \cdot \|_p$ denotes the norm on $L^p(X, \mu)$. We also use the abbreviation $L^p(I)$ for $L^p(I, \mathrm{m})$ when I is a subinterval of $\mathbb{R}$.

1. Fix $\epsilon > 0$. Prove that

$$f(x) = \begin{cases} x^{2+\epsilon}\cos(1/x^2) & x \in (0,1] \\ 0 & x = 0 \end{cases}$$

is absolutely continuous on $[0,1]$.

2. (a) Let $A_1 \subset A_2 \subset A_3 \subset \cdots$ be measurable sets with respect to the measure μ and let $A = \bigcup_{n=1}^{\infty} A_n$. Prove that $\lim_{n\to\infty} \mu(A_n) = \mu(A)$ by using the additive property of measures.

   (b) Give a counterexample to the statement: "Let $B_1 \supset B_2 \supset B_3 \supset \cdots$ be measurable sets with respect to the measure μ and let $B = \bigcap_{n=1}^{\infty} B_n$. Then $\lim_{n\to\infty} \mu(B_n) = \mu(B)$."

   (c) What hypothesis can we add to the statement in part (b) to make it true?

3. On $[a,b]$, suppose you have a sequence of functions f_k of bounded variation that converge pointwise to a function f. Assume that all total variations of f_k's are bounded from above by some M. Prove that f is of bounded variation and that its total variation is less or equal M.

4. Let $f \in L^1(\mathbb{R})$ be a non-negative function. Prove that there exists $a \in \mathbb{R}$ such that

$$\int_{-\infty}^{a} f(x)\,dx = 2\int_{a}^{\infty} f(x)\,dx.$$

5. Let (X, M, μ) be a finite measure space, μ be a positive measure, and let $f \in L^1(X,\mu)$ be a non-negative function. Prove that

$$\lim_{n\to\infty} \int_X \frac{f^n}{1+f^n}\,d\mu = \mu(E) + \frac{1}{2}\mu(F),$$

where $E = \{x \in X : f(x) > 1\}$ and $F = \{x \in X : f(x) = 1\}$.

6. On $[0,1]\times[0,1]$, define a function f by

$$f(x,y) = \begin{cases} xy & \text{if } xy \text{ is rational} \\ 0 & \text{otherwise.} \end{cases}$$

Prove that f is Lebesgue integrable and find its integral.

7. Find
$$\lim_{n\to\infty}\int_0^n (1-x/n)^n x^{1/n} dx.$$
You may use that $\ln(1-t) \le -t$ for $t \in (0,1)$.

8. Prove that
$$\int_0^\infty \mathrm{e}^{-x}\sqrt{7+4\cos x}\, dx \le 3.$$
You may use that $\int_0^\infty \mathrm{e}^{-x}\cos x\, dx = 1/2$.