# Joint Program Exam in Real Analysis

# September 11, 2018

**Instructions:**

1. Print your student ID (but not your name) and the problem number on each page. Write on one side of each paper sheet only. Start each problem on a new sheet. Write legibly using a dark pencil or pen.

2. You may use up to three and a half hours to complete this exam.

3. The exam consists of 8 problems. All problems are weighted equally.

4. For each problem which you attempt try to give a complete solution and justify carefully your reasoning. Completeness is important: a correct and complete solution to one problem will gain more credit than two half solutions to other problems. Justify the steps in your solutions by referring to theorems by name, when appropriate, and by verifying the hypotheses of these theorems. You do not need to reprove the theorems you used.

5. $\mathbb{R}$ denotes the set of real numbers, $\mathbb{N}$ denotes the set of positive integers, $\mathrm{m}(A)$ refers to the Lebesgue measure of the set $A \subset \mathbb{R}^d$, "measurable" refers to Lebesgue measurable, and "a.e." means almost everywhere with respect to Lebesgue measure unless noted otherwise. Instead of $d\mathrm{m}$ we sometimes write $dx$, $dt$, etc. referring to the variable to be integrated. $L^p(X, \mu)$ denotes the Lebesgue space of order $p$ with respect to the positive measure $\mu$ and $\| \cdot \|_p$ denotes the norm on $L^p(X, \mu)$. We also use the abbreviation $L^p(I)$ for $L^p(I, \mathrm{m})$ when $I$ is a subset of $\mathbb{R}$.

1. Determine for each of the following statements whether it is true or false? Justify your answer.

   (a) If a function $f$ is non-negative and continuous on $[0,\infty)$ and $\int_0^\infty f(x)\,dx < \infty$, then $f(x) \to 0$ as $x \to \infty$.

   (b) Let $f : [0,\infty) \to \mathbb{C}$ be function. Suppose there are constants $M > 0$ and $\theta \geq 1$ such that $|f(x) - f(y)| \leq M|x-y|^\theta$ for all $x, y \in [0,\infty)$. Then $f(x) = f(0) + \int_0^x f'(s)\,ds$ for all $x \in [0,\infty)$.

   (c) If two continuous functions on $\mathbb{R}$ are equal almost everywhere, then they are the same function.

2. Prove that $\lim_{n\to\infty} \int_{(0,\infty)} x/(1+x^n)\,dx$ exists and find its value.

3. Prove the "Layer Cake Formula", i.e., show that

$$\|f\|_p^p = \int_0^\infty pt^{p-1}\mathrm{m}(\{x \in \mathbb{R}^n : |f(x)| > t\})\,dt$$

   whenever $f$ is a measurable function on $\mathbb{R}^n$ and $p \in [1,\infty)$.

4. Suppose $\alpha$ and $\beta$ are positive constants with $\alpha > \beta$ and define the function

$$f(x) = \begin{cases} x^\alpha \sin x^{-\beta} & \text{if } 0 < x \leq 1 \\ 0 & \text{if } x = 0. \end{cases}$$

   Show that $f$ is absolutely continuous on $[0,1]$.

5. Show that

$$\int_1^\infty \frac{\sqrt[4]{3+2x}}{x^3}dx \leq \sqrt[4]{54/125}.$$

6. Let $f \in L^1(\mathbb{R})$ and suppose that there is $M > 0$ such that $|f(x)| \leq M$ for almost every $x \in \mathbb{R}$. Let $g(x) = \frac{1}{2}e^{-|x|}$.

   (a) Define, for $n \in \mathbb{N}$ and $x \in \mathbb{R}$,

$$f_n(x) = \int_{\mathbb{R}} f(x - \frac{y}{n})g(y)\,dy$$

   and show that at each $x \in \mathbb{R}$ where $f$ is continuous, $f_n(x) \to f(x)$ as $n \to \infty$.

(b) Define

$$f_\epsilon(x) = \frac{1}{\epsilon}\int_{\mathbb{R}} f(z)g(\frac{x-z}{\epsilon})\,dz.$$

Show that for each fixed $\epsilon > 0$, $f_\epsilon$ is continuous at every $x \in \mathbb{R}$.

7. State the monotone convergence theorem and Fatou's lemma. Prove the latter using the former.

8. Let $f$ be a measurable function. Suppose that there is a positive number $M$ such that $\|f\|_p \le M$ for all $p \in (1,\infty)$. Show that $\|f\|_\infty \le M$.