Joint Program Exam in Real Analysis

September 2016

Instructions:

1. Print your student ID and the problem number on each page. Write on one side of each paper sheet only. Start each problem on a new sheet. Write legibly using a dark pencil or pen.

2. You may use up to three and a half hours to complete this exam.

3. The exam consists of 8 problems. All the problems are weighted equally. You need to do 7 of the 8 problems for full credit.

4. For each problem which you attempt try to give a complete solution. Completeness is important: a correct and complete solution to one problem will gain more credit than two half solutions to two problems. Justify the steps in your solutions by referring to theorems by name, when appropriate, and by verifying the hypotheses of these theorems. You do not need to reprove the theorems you used.

5. Throughout this exam all the integrals are Lebesgue integrals. $L^p(E)$ denotes the L^p-space with respect to Lebesgue measure on the Lebesgue measurable set E, with corresponding norm $\|\cdot\|_p$. The words 'measurable' and 'integrable' refer to Lebesgue measure. By m we denote Lebesgue measure on $\mathbb{R}$ or a subset of $\mathbb{R}$. Integration with respect to m is also denoted by dx.

1. Let E be measurable in $\mathbb{R}^n$ and A non-measurable in $\mathbb{R}^k$. State and prove a necessary and sufficient condition on E for measurability of $E \times A$ in $\mathbb{R}^{n+k}$.

2. (a) Show that any real-valued function of bounded variation on $[0,1]$ can be represented as a difference of two increasing functions.

   (b) Can the two functions in the difference be chosen strictly increasing? Prove or disprove.

3. Let
$$f_n(x) = \frac{1}{1 + x^{\frac{n}{2016+\ln n}}}, \quad x \geq 0.$$
   Find $\lim_{n\to\infty} \int_{[0,\infty)} f_n \, dm$ and provide full justification.

4. Suppose that $f : [0,1] \to \mathbb{C}$ is measurable, $1 \leq p < \infty$, and $x^d f(x) \in L^p(0,1)$ for every $d > 0$. Prove that $f \in L^p(0,1)$ if and only if there exists a finite constant c such that $\int_0^1 |x^d f(x)|^p \, dx \leq c$ for all $d > 0$.

5. Characterize all measurable subsets E of $\mathbb{R}$ for which the following holds: All measurable functions $f : E \to \mathbb{C}$ are square integrable with respect to Lebesgue measure.

6. Suppose that $f : \mathbb{R} \to \mathbb{C}$ is measurable and define $g(x) = xf(x)$. If $f, g \in L^2(\mathbb{R})$, then prove that $f \in L^1(\mathbb{R})$ and
$$\|f\|_1 \leq \sqrt{2}(\|f\|_2 + \|g\|_2).$$

7. Suppose μ is a finite Borel measure on $\mathbb{R}$ and define $F(x) = \mu((-\infty, x])$. Prove that for any $c \in \mathbb{R}$,
$$\int_{\mathbb{R}} (F(x+c) - F(x)) \, dx = c\mu(\mathbb{R}).$$

8. Let $\mathbb{N}$ be the positive integers and $\mathcal{M}$ the σ-algebra of all subsets of $\mathbb{N}$. For a sequence $b_n \geq 0$, $n \in \mathbb{N}$, a measure μ_b on $(\mathbb{N}, \mathcal{M})$ is given by $\mu_b(E) = \sum_{n\in E} b_n$. Let $c_n \geq 0$, $n \in \mathbb{N}$, be another such sequence with associated measure μ_c.

   (a) Give a necessary and sufficient criterion for μ_c to be absolutely continuous with respect to μ_b.

   (b) What is $d\mu_c/d\mu_b$ in this case?