Joint Program Exam in Real Analysis

September 15, 2015

Instructions:

1. Print your student ID and the problem number on each page. Write on one side of each paper sheet only. Start each problem on a new sheet. Write legibly using a dark pencil or pen.

2. You may use up to three and a half hours to complete this exam.

3. The exam consists of 8 problems. All problems are weighted equally. You need to do all of the problems for full credit.

4. For each problem which you attempt try to give a complete solution and justify carefully your reasoning. Completeness is important: a correct and complete solution to one problem will gain more credit than two half solutions to two problems. Justify the steps in your solutions by referring to theorems by name, when appropriate, and by verifying the hypotheses of these theorems. You do not need to reprove the theorems you used.

5. $\mathbb{R}$ denotes the set of real numbers, $\mathbb{N}$ denotes the set of positive integers, $\mathrm{m}(A)$ refers to the Lebesgue measure of the set $A \subset \mathbb{R}^d$, "measurable" refers to Lebesgue measurable, and "a.e." means almost everywhere with respect to Lebesgue measure. Instead of $d\mathrm{m}$ we sometimes write dx, dt, etc. referring to the variable to be integrated. $L^p(X, \mu)$ denotes the Lebesgue space of order p with respect to the positive measure μ and $\| \cdot \|_p$ denotes the norm on $L^p(X, \mu)$.

1. Suppose $f : [0, \infty) \to [0, \infty)$ is measurable and satisfies $\int_1^\infty f \, dm < \infty$ (note that 1 is in the lower bound of the integral). Prove that

$$\lim_{n\to\infty} \int_0^\infty \frac{f^n}{2+f^n} dm = \frac{1}{3}\mu(E) + \mu(F)$$

where $E = \{x : f(x) = 1\}$ and $F = \{x : f(x) > 1\}$.

2. Define

$$g(x) = \begin{cases} x^p \cos(1/x) & \text{for } x \in (0, 1], \\ 0 & \text{for } x = 0. \end{cases}$$

Show that g is absolutely continuous on $[0, 1]$ if $p > 1$.

3. Suppose A is a set of positive measure in $[0, 1]$. Prove that there are two elements of A that differ by a rational number other than zero.

4. Assume that for some $M > 0$ and all $p \in (1, \infty)$ we have $\|f\|_p \le M$. Show that $\|f\|_\infty \le M$.

5. Let $E \subseteq (0, 2\pi)$ be a measurable set. Show that

$$\lim_{k\to\infty} \int_E \cos(kx) \, dx = 0 \quad \text{and} \quad \lim_{k\to\infty} \int_E (\cos(kx))^2 \, dx = \frac{1}{2}\mathrm{m}(E).$$

6. Show that

$$\int_1^\infty \sqrt[3]{1+x^3} x^{-4} dx \le \sqrt[3]{7/40}.$$

7. Let

$$(f \star g)(x) = \int_0^\infty f\left(\frac{x}{t}\right) g(t) \frac{dt}{t}.$$

Prove that if $f \in L^p((0, \infty), \mu)$ and $g \in L^1((0, \infty), \mu)$, then $f \star g \in L^p((0, \infty), \mu)$, where μ is the measure defined by $\mu(A) = \int_A (1/x) \, dx$.

8. Let $f \in L^1((0, 1), \mathrm{m})$ and suppose that $\lim_{x\uparrow 1} f(x) = A$. Show that

$$\lim_{n\to\infty} \int_0^1 n x^n f(x) dx = A.$$