Joint Program Exam in Real Analysis

September 2012

Instructions:

1. You may use up to three and a half hours to complete this exam.
2. The exam consists of 7 problems. You need to do ALL of them for full credit.
3. For each problem which you attempt try to give a complete solution. Completeness is important: a correct and complete solution to one problem will gain more credit than two "half solutions" to two problems. Justify the steps in your solutions by referring to theorems by name, when appropriate, and by verifying the hypotheses of these theorems. You do not need to reprove the theorems you used.
4. Throughout the exam all the integrals mean the Lebesgue integrals. $L^p(E)$ denotes the L^p space with respect to Lebesgue measure on the Lebesgue measurable set E.
5. Begin each solution on a new page and write the last four digits of your university student ID number, and problem number, on every page. Please write only on one side of each sheet of paper.
6. The use of calculators or other electronic gadgets is not permitted during the exam.
7. Write legibly using dark pencil or pen.

1. Are the following statements true or false? Justify! If a statement is false, provide a counter-example.

   a) If $f_k \in L(0,1)$ for $k = 1, 2, \cdots$, and $f_k \to 0$ uniformly on every closed interval in $(0,1)$, then $\int_0^1 |f_k(x)|\,dx \to 0$ as $k \to \infty$.

   b) If $f_k \in L(0,1)$ for $k = 1, 2, \cdots$, and $\int_0^1 |f_k(x)|\,dx \to 0$ as $k \to \infty$, then $f_k \to 0$ almost everywhere in $(0,1)$.

2. Prove the following statements:

   a) The function $f(x) = \dfrac{1}{x} \sin \dfrac{1}{x}$ is not Lebesgue integrable on $(0, \pi)$.

   b) Let E be a Lebesgue measurable set in $\mathbb{R}^n$ and f be a Lebesgue measurable function on E. If $\int_A f(x)\,dx = 0$ for every compact subset A of E, then $f = 0$ almost everywhere in E.

3. Do the following:

   a) Let X be an uncountable set, let M be a collection of all sets $E \subset X$ such that either E or its complement E^c is at most countable. Prove that M is a σ-algebra in X.

   b) Let $f : \mathbb{R} \to \mathbb{R}$ be a non-decreasing function (f is not necessarily continuous). Prove that f is Borel measurable.

4. Find the following limit and justify your answer:

$$\lim_{n\to\infty} \int_0^\infty \frac{n x^{\frac{1}{n}}}{2ne^x + \sin(nx)}\,dx.$$

5. Find all the nonnegative functions $g \in L^3(0,1)$, satisfying the equation:

$$\left(\int_0^1 xg(x)\,dx\right)^3 = \frac{4}{25}\int_0^1 g^3(x)\,dx.$$

6. Let

$$f(x) = \begin{cases} x(\ln x)^2, & x \in (0,1], \\ 0, & x = 0. \end{cases}$$

Prove that f is absolutely continuous on $[0,1]$.

7. Let $f \in L(0,1)$. Prove that

(i) for every $x \in (0,1)$,

$$g(x) = \int_x^1 \frac{f(t)}{t}\,dt$$

exists and is finite;

(ii) $g \in L(0,1)$ and $\int_0^1 g(x)\,dx = \int_0^1 f(x)\,dx$.