# Joint Program Exam in Real Analysis

# May 2012

**Instructions:**

1. You may use up to three and a half hours to complete this exam.
2. The exam consists of 7 problems. All the problems are weighted equally. You need to do ALL of them for full credit.
3. For each problem which you attempt try to give a complete solution. Justify the steps in your solutions by referring to theorems by name, when appropriate, and by verifying the hypotheses of these theorems. You do not need to reprove the theorems you used. Completeness is important: a correct and complete solution to one problem will gain more credit than two "half solutions" to two problems.
4. Throughout the exam:

   (1). $m$ denotes the Lebesgue measure. "Measurable" is short for "Lebesgue-measurable". Instead of $dm$ we sometimes write $dx$ or $dy$, referring to the variable to be integrated.

   (2). $L^p(a,b)$ denotes the $L^p$ space with respect to $m$ on the interval $(a,b)$, $-\infty \le a < b \le \infty$.

   (3). $\|\cdot\|_p$ $(0 < p \le \infty)$ denotes $L^p$-norm.

   (4). $V_a^b(f)$ denotes the variation of $f$ over a closed bounded interval $[a,\, b]$.

1. Are the following statements true or false? Justify! If a statement is false, provide a counter-example.

   a) Let $\{E_k\}_{k\in\mathbb{N}}$ be a sequence of measurable subsets of $[0, 1]$ satisfying $m(E_k) = 1$ for $k \in \mathbb{N}$. Then

$$m\left(\bigcap_{k=1}^{\infty} E_k\right) = 1.$$

   b) Let $f, g : \mathbb{R} \to \mathbb{R}$ be continuous and $f = g$ a.e. on $\mathbb{R}$. Then, $f(x) = g(x)$ for all $x$.

2. Are the following statements true or false? Justify! If a statement is false, provide a counter-example.

   a) If $f \in L^p(0, 1)$ for every $p \in (1, \infty)$, then $f \in L^\infty(0, 1)$.

   b) If $p < q$ for some $p, q \in [1, \infty)$, then $L^q[1, \infty) \subseteq L^p[1, \infty)$.

3. Find

$$\lim_{n\to\infty} \int_{[0,\, n]} \left(1 + \frac{x}{n}\right)^n e^{-\pi x}\, dx.$$

   Justify your answer!

4. Let $f$ be Borel measurable and $f \in L^1(\mathbb{R})$. Prove that, for a.e. $x \in \mathbb{R}$,

$$\varphi(x) = \int_{(0,\,1)} \frac{f(x-y)}{\sqrt{y}} dy$$

   exists, is finite and $\|\varphi\|_1 \le 2\|f\|_1$.

5. Let $g : [a, b] \to \mathbb{R}$ be absolutely continuous and $f(x) = \sin(g(x))$. Prove that

   (a) $V_a^b(f) \leq V_a^b(g)$.

   (b) The function $f(x)$ is absolutely continuous.

6. Let $p \in [1, \infty)$, and suppose that $f_n \in L^p[0, 1]$ converges a.e. to $f \in L^p[0, 1]$. Show that $f_n$ converges to $f$ in $L^p[0, 1]$ if and only if $\|f_n\|_p \to \|f\|_p$.

7. Let $f \in L^\infty[0, 1]$ and $\|f\|_\infty \leq 1$.

   (a) Show that

$$\int_{[0,1]} \sqrt{1 - f^2(x)}dx \leq \sqrt{1 - \left(\int_{[0,1]} f(x)dx\right)^2}.$$

   (b) Describe the class of functions for which the equality takes place.