Joint Program Exam in Real Analysis

Fall 2008 2008

Instructions

1. You may take up to $3\frac{1}{2}$ hours to complete the exam.
2. Work seven out of the eight problems. Completeness in your answers is very important. An essentially complete and correct solution to one problem with gain more credit than solutions to two problems, each of which is "half correct".
3. All of the numbered problems have equal weight.
4. When appropriate, refer to a theorem by name or by providing a brief theorem statement. You do not need to reprove theorems you used in your proof.

Notation

Throughout the exam, $\mathbb{R}$ stands for the set of real numbers, $\mathbb{Z}$ for the set of integers. Lebesgue measure on $\mathbb{R}$ is denoted by m and on $\mathbb{R}^2$ by m_2. On the real line, notation such as $\int_{[0,1]} f$, $\int_{[0,1]} f\,dm$, $\int_{[0,1]} f(x)dx$, etc. is used for Lebesgue integrals, while Riemann integrals are denoted $\int_0^1 f(x)dx$, $\int_0^\infty f(x)dx$. By $V_a^b(f)$ we denote the variation of function f on interval $[a,b]$.

By $L^p(a,b)$, $p \geq 0$, we denote a class of complex Lebesgue measurable functions f on (a,b) such that

$$\int_{(a,b)} |f|^p \; dm < \infty.$$

Spaces $L^p(\mathbb{R})$, $L^p(\mathbb{R}^2)$ are defined in the analogous way.

1. Find the Lebesgue measure of the set of algebraic numbers in $\mathbb{R}$. Recall that r is an algebraic number if and only if there is a polynomial $p(x)$ with all coefficients in $\mathbb{Z}$ such that $p(r) = 0$ (we assume that $p(x)$ has at least one nonzero coefficient).

2. Prove that the function
$$f(x) = \begin{cases} \sqrt{x}\ln x & \text{if } x \in (0,1] \\ 0 & \text{if } x = 0 \end{cases}$$
is absolutely continuous on $[0, 1]$.

3. Let $f_n, n = 1, 2, 3...$ be a sequence of nonnegative continuous functions on $\mathbb{R}$ such that
$$\int_{\mathbb{R}} f_n dm < \frac{1}{n^3}.$$
Let $f(x) = \sum_{n=1}^{\infty} f_n(x)$. Prove that $f(x)$ is integrable on $\mathbb{R}$.

4. Let $f \in L^1(0, \infty)$. Prove that there is a sequence $x_n \to \infty$ such that $\lim_{n\to\infty} x_n f(x_n) = 0$.

5. Find all the functions $g(x) \in L^3(0, 1)$, satisfying the equation:
$$\left(\int_{[0,1]} xg(x)dx\right)^3 = \frac{4}{25}\int_{[0,1]} g^3(x)dx.$$

6. Find $V_{-1}^2(f)$ if $f(x) = x(x^2 - 1)$.

7. Find the limit:
$$\lim_{n\to\infty} \int_0^{\infty} \frac{nx^{\frac{1}{n}}}{2ne^x + \sin(nx)} dx$$

8. Let $K \in L^2(\mathbb{R}^2)$, $f \in L^2(\mathbb{R})$. For each $x \in \mathbb{R}$, at which the integral exists, set
$$g(x) = \int_{\mathbb{R}} K(x, y)f(y)dy.$$
Prove that $g \in L^2(\mathbb{R})$.