Joint Program Exam in Real Analysis

September 11, 2007

Instruction

You may take up to three and a half hours to complete the exam. **Work seven out of the eight problems**. Completeness in your answers is very important. An essentially complete and correct solution to one problem will gain more credit than solutions to two problems with each of which is "half correct". When appropriate, justify your steps by referring the name(s) of theorem(s) used, or by providing a brief theorem statement(s). You do not need to reprove the theorems used in your proof.

Notation

Throughout the exam, $\mathbb{R}$ stands for the set of all real numbers. Notation such as $\int_{[0,1]} f$, $\int_{[0,1]} f\,dm$ and $\int_{[0,1]} f(x)\,dx$, etc. are used for the Lebesgue integrals.

Problem 1 *Suppose $f:\ \mathbb{R} \to \mathbb{R}$ is a continuous and integrable function. Is it true that $\lim_{x\to\infty} f(x) = 0$? (Prove or give a counterexample.)*

Problem 2 *Given that $\int_{[0,\infty)} e^{-x}\sin x\,dx = \frac{1}{2}$, prove that*

$$\int_{[0,\infty)} e^{-x}\sqrt{8 + 2\sin x}\,dx \le 3\,.$$

Problem 3 *Does there exist an absolutely continuous function f on $[0,1]$ and a sequence of Borel sets $E_n \subset [0,1]$, $n = 1, 2, \cdots$, such that $m(E_n) > 0$ and*

$$\frac{m\left(f(E_n)\right)}{m(E_n)} > n$$

for every $n \ge 1$?

Problem 4 *Let E be a measurable subset of $\mathbb{R}^n$. Suppose there exists a number $c \in (0,1)$ such that $m(E\cap B) < c\,m(B)$ for every n-dimensional rectangle (box) $B \subset \mathbb{R}^n$. Prove that $m(E) = 0$.*

Problem 5 *Let $f \in L^1(\mathbb{R}^2)$. Prove that there exist a subset $E \subset \mathbb{R}^2$ such that*

$$\int_E f\,dm = \int_{\mathbb{R}^2\setminus E} f\,dm.$$

Problem 6 *Let E be a measurable subset of $\mathbb{R}^n$ and $m(E) > 0$. Suppose that f is in both $L^p(E)$ and $L^r(E)$, where $p, r \ge 1$. Prove that f is in $L^{(p+r)/2}(E)$.*

Problem 7 *Let f_n, $n = 1, 2, 3, \cdots$ be a sequence of nonnegative continuous functions on $\mathbb{R}$ such that $\int_{\mathbb{R}} f_n \le \frac{1}{n^3}$. Let $f = \sum_{n=1}^{\infty} f_n$. Prove that f is integrable on $\mathbb{R}$.*

Problem 8 *Show that $f_n(x) = e^{-n|1-\sin x|}$ converges in measure to $f(x) = 0$ on $[a,b] \subset \mathbb{R}$.*