Joint Program Exam, September 2004

Real Analysis

Instructions. You may use up to 3.5 hours to complete this exam. For each problem which you attempt try to give a complete solution. Completeness is important: a correct and complete solution to one problem will gain more credit than two "half solutions" to two problems.

Justify the steps in your solutions by referring to theorems by name, when appropriate, and by verifying the hypotheses of these theorems. You do not need to reprove the theorems you used.

Throughout this exam m denotes Lebesgue measure on $\mathbb{R}$. 'Measurable' is short for 'Lebesgue-measurable'. Instead of dm we sometimes write dx or dy, referring to the variable to be integrated. $L^p(a,b)$ is the L^p space with respect to m on the interval (a,b).

Part 1 below accounts for 40 percent of the exam grade, Part 2 for 60 percent. Separately the questions of Parts 1 and 2 carry equal weight.

Part 1.

DO ALL PROBLEMS IN PART ONE.

Are the following statements true or false? Justify!

1. There is a sequence of measurable subsets E_n of $\mathbb{R}$ with $E_n \subset E_{n+1}$ for $n = 1, 2, \ldots$, such that $m(\cup_n E_n) \neq \lim_{n\to\infty} m(E_n)$.
2. There is a sequence of measurable subsets D_n of $\mathbb{R}$ with $D_n \supset D_{n+1}$ for $n = 1, 2, \ldots$, such that $m(\cap_n D_n) \neq \lim_{n\to\infty} m(D_n)$.
3. There are measurable functions f_n, $n = 1, 2, \ldots$, and f on $[0, 1]$ such that $f_n(x) \to f(x)$ for every $x \in [0, 1]$, but $\int_{[0,1]} f_n \, dm \not\to \int_{[0,1]} f \, dm$.
4. There is a subset A of $\mathbb{R}$ which is not Lebesgue measurable, but such that $B = \{x \in A : x \text{ is irrational}\}$ is Lebesgue measurable.
5. There exists an absolutely continuous function f on $[0, 1]$ such that $f(0) = 0$, $f(1) = 1$, and $f'(t) = 0$ for almost every $t \in [0, 1]$.

Part 2.

DO 4 PROBLEMS IN PART TWO. MARK THE ONES TO BE GRADED.

1. Let $f : [0,1] \to \mathbb{R}$ be defined by
$$f(x) = \left\{ \begin{array}{ll} \sqrt{x} & \text{if } x \text{ is irrational} \\ 0 & \text{otherwise.} \end{array} \right\}$$
   (i) Show that f is measurable.

   (ii) Is f Lebesgue integrable? If yes, find its Lebesgue integral.

   (iii) Is f Riemann integrable? If yes, find its Riemann integral.

2. Suppose that $f_n,\ g_n,\ f,\ g \in L^1(\mathbb{R})$, $f_n \to f$, $g_n \to g$ almost everywhere in $\mathbb{R}$, and $\int_{\mathbb{R}} g_n\, dm \to \int_{\mathbb{R}} g\, dm$ as $n \to \infty$. If $|f_n| \leq g_n$ for all n, prove that
$$\lim_{n\to\infty} \int_{\mathbb{R}} f_n\, dm = \int_{\mathbb{R}} f\, dm.$$

3. Prove that
$$\int_0^1 \sqrt{x^4 + 4x^2 + 3}\, dx \leq \frac{2}{3}\sqrt{10}.$$

4. Prove or disprove: There is a function f on $(0,1)$ such that $f \in L^p(0,1)$ for all $p \in [1,\infty)$, but $f \notin L^\infty(0,1)$.

5. Let $f \in L^1(0,1)$, $f \geq 0$. Show that

   (i) $\displaystyle\int_0^1 \frac{f(y)}{|x-y|^{1/2}}\, dy < \infty$ for almost every $x \in [0,1]$,

   (ii) $\displaystyle\int_0^1 \frac{f(y)^{1/2}}{|x-y|^{1/4}}\, dy < \infty$ for every $x \in [0,1]$.

6. (i) Let f and g be absolutely continuous on $[0,1]$. Show that fg is absolutely continuous and that
$$(fg)'(x) = f'(x)g(x) + f(x)g'(x) \quad \text{for almost every } x \in [0,1].$$
   (ii) Let g be absolutely continuous on $[0,1]$. Show that there is a finite constant C (only depending on g) such that
$$\left| \int_0^1 \sin(kx)g(x)\, dx \right| \leq \frac{C}{|k|}$$
   for all non-zero k.