Joint Program Exam of May, 2003

in Real Analysis

Instructions:

You may take up to three and a half hours to complete this exam.

Work 7 out of the 9 problems. Full credit can be gained with 7 essentially complete and correct solutions.

Justify each of your steps by referring to theorems by name where appropriate, or by providing a brief theorem statement. You do not need to reprove the theorems you use.

For each problem you attempt, try to give a complete solution. A correct and complete solution to one problem will gain more credit than solutions to two problems, each of which is "half-correct".

Notation:

$\mathbb{R}$ denotes the set of real numbers, $m(E)$ refers to the Lebesgue measure of the set $E \subset \mathbb{R}$, "measurable" refers to Lebesgue measure and "a.e." means almost everywhere with respect to Lebesgue measure.

Problem 1.

Give an example or prove non-existence of such.

(a) A subset of $\mathbb{R}$ of measure zero, whose closure has positive measure.

(b) A sequence (f_n) of functions in $L^1[0,1]$ such that $f_n \to 0$ pointwise and yet $\int_{[0,1]} f_n \, dm \to \infty$.

Problem 2.

(a) Let E be a measurable subset of $\mathbb{R}^2$. Suppose that, for a.e. $x \in \mathbb{R}$, the set $E_x \overset{def}{=} \{y \in \mathbb{R} : (x,y) \in E\}$ has measure zero in $\mathbb{R}$. Prove that, for a.e. $y \in \mathbb{R}$, the set $E^y \overset{def}{=} \{x \in \mathbb{R} : (x,y) \in E\}$ has measure zero in $\mathbb{R}$.

(b) Let A be a non-measurable subset of $\mathbb{R}^2$ whose intersection with the y-axis is not empty. Can the set $A_0 \overset{def}{=} \{y \in \mathbb{R} : (0,y) \in A\}$ be measurable for some such A?

Problem 3.

Let $f \in L^1(\mathbb{R}) \cap L^{17}(\mathbb{R})$. Prove that $f \in L^5(\mathbb{R})$.

Problem 4.

Let $E = [0,\infty)$. Prove that $\lim_{n\to\infty} \int_E \frac{x}{1+x^n} \, dx$ exists, and find its value. Justify all your assertions.

Problem 5.

Let E be a measurable subset of $\mathbb{R}$, and let $f, f_k \in L^1(E)$, $k \in \mathbb{N}$. Suppose that $f_k \to f$ a.e. on E and $\|f_k\|_1 \to \|f\|_1$. Prove that then $f_k \to f$ in $L^1(E)$.

Problem 6.

Let $f \in L^1[0,1]$. Prove that, for a.e. $x \in [0,1]$, $\int_{[0,1]} \frac{f(y)}{\sqrt{|x-y|}} \, dm(y)$ exists and is finite.

Problem 7.
Let f be continuous and strictly increasing on $[0,1]$. Suppose that $m\big(f(E)\big) = 0$ for every set $E \subset [0,1]$ with $m(E) = 0$. Show that f is absolutely continuous.

Problem 8.
Let f be integrable on $[0,1]$. Prove that there exists $c \in [0,1]$ such that $\int_{[0,c]} f\,dm = \int_{[c,1]} f\,dm$.

Problem 9.
Let f be a Lebesgue measurable function on $\mathbb{R}$. Show that:

$$\int_{\mathbb{R}} |f|^3\,dm = 3\int_0^\infty t^2 m\big(\{|f| > t\}\big)\,dt.$$