Joint Program Exam in Real Analysis
September 14, 1999

Instructions: You may take up to $3\frac{1}{2}$ hours to complete the exam. Completeness in your answers is very important. Justify your steps by referring to theorems by name when appropriate. An essentially complete and correct solution to one problem will gain more credit than solutions to two problems each of which is "half correct".

Notation: Throughout the exam the symbol $\mu(E)$ refers to Lebesgue measure of the set E, and for simplicity, $d\mu(x)$ will be denoted by dx. Also the symbols "$\mathbb{Q}$" and "$\mathbb{R}$" stand for the rational and real numbers respectively.

1. Let $f : \mathbb{R} \to \mathbb{R}$ be measurable. Define $h : \mathbb{R} \to \mathbb{R}$ by

$$h(x) = \begin{cases} 0 & \text{if } f(x) \in \mathbb{Q} \\ x & \text{otherwise.} \end{cases}$$

(a) Show that the set $E = \{x \in \mathbb{R} : f(x) \in \mathbb{Q}\}$ is measurable.

(b) Show that h is measurable.

2. Let f be nonnegative and integrable on $\mathbb{R}$. Define $\phi : (0, \infty) \to \mathbb{R}$ by

$$\phi(t) = \mu(\{x : f(x) \geq t\}).$$

Show that

(i) if $0 < a < b < \infty$, then ϕ is of **bounded variation** on $[a, b]$.

(ii) $\lim_{t\to\infty} t\phi(t) = 0$.

(iii) Is ϕ necessarily **absolutely continuous** on $[a, b]$? (why?)

3. Let $E = [0, \infty)$. Prove that $\lim_{n\to\infty} \int_E \frac{x}{1+x^n} dx$ exists, and find its value. Justify all your assertions.

4. Let $E = [0, \infty)$. If $f \geq 0$ on E and $f \in L^1(E)$ then prove or disprove that

$$\lim_{x\to\infty} f(x) = 0.$$

5. Let $E = [1, \infty)$ and suppose $f \in L^2(E)$ such that $f \geq 0$ a.e. in E. Define

$$g(x) = \int_E f(t)\mathrm{e}^{-tx} dt.$$

Show that $g \in L^1(E)$ and

$$2\|g\|_1 \leq \|f\|_2.$$

6. Suppose

(i) $f_n, f \in L^1[0, 1]$;

(ii) $f_n \to f$ a.e. in $[0, 1]$;

(iii) $\|f_n\|_1 \to \|f\|_1$.

Show that $\|f_n - f\|_1 \to 0$ as $n \to \infty$.

7. Let $f \in L^1[0, 1]$ and $f \geq 0$. Show that $\int_0^1 \frac{f(y)}{|x-y|^{2/3}} dy$ is finite for a.e. $x \in [0, 1]$.

8. Let $f \in L^1(\mathbb{R}) \cap L^3(\mathbb{R})$. Prove that $f \in L^2(\mathbb{R})$.