JOINT PROGRAM EXAM
REAL ANALYSIS
FALL 1998

Instructions: You may take up to $3\frac{1}{2}$ hours to complete the exam. Completeness in your answers is very important. Justify your steps by referring to theorems by name when appropriate, or by providing a brief theorem statement. An essentially complete and correct solution to one problem will gain more credit than solutions to two problems each of which is half correct.

Notation: Throughout the exam the symbol "$\mu(E)$" refers to Lebesgue measure of the set E. Also the symbol "$\mathbb{R}$" stands for the real numbers. When dx is used in integration, the measure will be the Lebesgue measure on the real numbers.

DO THE FOLLOWING 8 PROBLEMS

Note: Make sure that you carefully justify your claims.

1. Let E be a measurable subset of $\mathbb{R}$. Suppose there exists a number $c \in (0,1)$ such that

$$\mu(E \cap I) \leq c\mu(I)$$

for all intervals I in $\mathbb{R}$. Show that $\mu(E) = 0$.

2. Evaluate the following limit and justify each of your steps.

$$\lim_{n\to\infty} \int_1^\infty \frac{\ln(1+nx)}{1+x^2 \ln n}\, dx.$$

3. Let $f \in L^1(\mathbb{R})$ and $g \in L^\infty(\mathbb{R})$. Define

$$h(x) = \int_{\mathbb{R}} f(x-t)g(t)dt.$$

Show that

$$\|h\|_1 \leq \|f\|_1 \|g\|_\infty.$$

4. Let $f, \{f_k\} \in L^2$. Suppose (i) $f_k \to f$ a.e. and (ii) $\|f_k\|_2 \to \|f\|_2$. Show that

$$\|f - f_k\|_2 \to 0 \quad \text{as} \quad k \to \infty.$$

5. Let $f \in L^1(0,\infty)$. Suppose $f > 0$ a.e. Define $F(t) = t^{-1}\int_0^t f(x)dx$. Prove that $F \notin L^1(0,\infty)$.

6. Prove or disprove whether there exists a sequence $\{f_n\}$ in $L^1[0,1]$ such that
(i) $|f_n(x)| \le 2$ for all $x \in [0,1]$ and all $n \in \mathbb{N}$;
(ii) $f_n(x) \to 0$ a.e. and
(iii) $\lim_{n\to\infty} \int_0^1 f_n(x)dx = 1/2$.

7. If $\int_A f = 0$ for every measurable subset A of a measurable set E, show that

$$f = 0 \text{ a.e. in } E.$$

8. Let $f(t,x)$ be defined on $[0,1]\times\mathbb{R}$ such that $f(t,\cdot) \in L^1(\mathbb{R})$. Suppose there exists $g \in L^1(\mathbb{R})$ such that

$$|f(t,x)| \le g(x) \quad \text{for all} \quad (t,x) \in [0,1] \times \mathbb{R}.$$

If $t_0 \in [0,1]$ and, for almost every $x \in \mathbb{R}$, $f(t,x)$ is continuous at t_0, then show that

$$F(t) = \int_{\mathbb{R}} f(t,x)dx$$

is also continuous at t_0.