Joint Program Examination in Real Analysis

May 1998

Instructions: You may use up to $3\frac{1}{2}$ hours to complete this exam. For each problem which you attempt, try to give a complete solution. Completeness is important: a correct and complete solution to one problem will gain more credit than "half correct" solutions to two problems.

Justify the steps in your solution by referring to theorems by name, when appropriate. (If you cannot remember the name of a theorem but do remember the statement, you may give the statement instead). When you refer to named theorems (e.g., Fubini's Theorem or the Monotone Convergence Theorem) in a problem, you do not need to reprove the named theorem.

Throughout this examination, the symbol $\mathbb{R}$ denotes the real numbers. When dx is used in integration, the measure will be Legesgue measure on the real numbers.

Part I.

For each problem below, a correct solution consists either of an example, as called for by the problem, or an explanation why no such example exists. The explanation should generally refer to a known theorem.

DO 4 OF THE PROBLEMS IN PART ONE.

1. A sequence, f_n, of functions in $L^1[0,1]$ such that $|f_n(x)| \leq 1$ for all x and for all n, $\lim_{n\to\infty} f_n(x) = 0$ for all x, and $\lim_{n\to\infty} \int_0^1 f_n(x)\,dx = 1$.

2. A sequence, f_n, of functions in $L^1[0,\infty]$ such that $|f_n(x)| \leq 1$ for all x and for all n, $\lim_{n\to\infty} f_n(x) = 0$ for all x, and $\lim_{n\to\infty} \int_0^\infty f_n(x)\,dx = 1$.

3. A subset S of $\mathbb{R}$ with Lebesgue measure 0 whose closure $\overline{S}$ has non-zero measure.

4. A bounded Lebesgue measurable function f on $[0,1]$ which is <u>not</u> Riemann integrable.

5. A sequence (x_n) which is in $\ell^{1+\epsilon}$, for all $\epsilon > 0$, but which is not in ℓ^1.

6. An absolutely continuous monotonic function defined on $[0,1]$ such that $f(0) = 0$, $f(1) = 1$, and $f'(t) = 0$ a.e. on $[0,1]$.

Part II.

Try to give a complete solution for each of the problems which you attempt in this part.

DO 5 OF THE PROBLEMS IN PART TWO.

1. Let $1 \leq p < q < \infty$. Show that $L^q[0,1]$ is a proper subset of $L^p[0,1]$. (Be sure to show that the containment is strict.)

2. A function f defined on $\mathbb{R}$ is said to satisfy a *Lipschitz condition* if there is a constant M such that, for all $x, y \in \mathbb{R}$,
$$|f(x) - f(y)| \leq M|x - y|.$$
Prove that if f satisfies a Lipschitz condition, then there exist monotone functions h and g such that $f(x) = h(x) - g(x)$, for all x.

3. Let $A \subset [0,1]$ be a non-measurable set. Let $B = \{(x,0) \in \mathbb{R}^2 \mid x \in A\}$.

   (a) Is B a Lebesgue measurable subset of $\mathbb{R}^2$? Prove your answer.

   (b) Can B be a closed subset of $\mathbb{R}^2$ for some such A? Prove your answer.

4. Evaluate the following limit and justify your calculations.

$$\lim_{n\to\infty} \int_0^\infty \frac{n \sin(x/n)}{x(1+x^2)}\, dx.$$

5. Let $(X, \mathcal{M}, \mu)$ be a measure space with $\mu(X) < \infty$. Suppose that $f\colon X \to \mathbb{R}$ is measurable and satisfies $\|f\|_\infty < \infty$. Show that

$$\lim_{p\to\infty} \|f\|_p = \|f\|_\infty.$$

6. Prove or disprove each of the following:

   (a) Let $-\infty < a < b < \infty$ and let f and g be absolutely continuous on $[a,b]$. Then fg is absolutely continuous on $[a,b]$.

   (b) Let f and g be absolutely continous on $\mathbb{R}$. Then fg is absolutely continuous on $\mathbb{R}$.

7. Let f and g be integrable functions defined on $\mathbb{R}$. Assume that $f(t) \geq 0$ and $g(t) \geq 0$, for all $t \in \mathbb{R}$. The *convolution* of f and g is defined as follows:

$$f * g(t) = \int_{\mathbb{R}} f(t-x) g(x)\, dx.$$

   Prove that $f * g$ is integrable and that $\|f * g\|_1 = \|f\|_1 \, \|g\|_1$

8. Let $f\colon \mathbb{R}^2 \to \mathbb{R}$ be defined by

$$f(x,y) = 2(x-y)e^{-(x-y)^2} u(x), \quad \text{where } u(x) = \begin{cases} 1, & \text{if } x > 0 \\ 0, & \text{if } x \leq 0. \end{cases}$$

   You may use without calculation the following two facts: $\displaystyle\int_{\mathbf{R}} \left[\int_{\mathbf{R}} f(x,y)\, dy \right] dx = 0$ and $\displaystyle\int_{\mathbf{R}} \left[\int_{\mathbf{R}} f(x,y)\, dx \right] dy = \sqrt{\pi}$.

   Determine the value of the integral: $\displaystyle\int_{\mathbf{R}^2} |f(x,y)|\, dx\, dy$. Justify your answer.

9. Show that

(a) $x^{-1}\sin x \notin L^1[0,\infty]$, that is

$$\int_0^\infty \frac{|\sin x|}{x}\,dx = \infty.$$

(b)

$$\lim_{A\to\infty}\int_0^A \frac{\sin x}{x}\,dx \quad \text{exists and is finite.}$$