JOINT PROGRAM EXAM
REAL ANALYSIS
May 1997

Instructions: You may take up to three and a half hours to complete the exam. Completeness in your answers is very important. Justify **each** of your steps by referring to theorems by name if appropriate or by providing a brief statement of the theorem.

An essentially complete and correct solution to one problem will gain more credit than solutions to two problems each of which is "half correct".

Full credit can be gained with 8 essentially complete and correct solutions.

Notation: By $L^p(\mu)$ we denote the Lebesgue spaces associated with a measure space (X, μ). Lebesgue measure on $\mathbb{R}$ and its subsets is denoted by m. If X is a subset of $\mathbb{R}$ and $\mu = m$ we use the notation $L^p(X)$ instead of $L^p(\mu)$. We write ℓ^p for the Lebesgue spaces associated with the counting measure on $\mathbb{N}$.

1. Give an example for each of the following objects or a brief explanation why it does not exist.
   (a) A set A of real numbers which is not Lebesgue measurable but for which the set $B = \{x \in A : x \text{ is irrational}\}$ is Lebesgue measurable.
   (b) A function on $\mathbb{R}$ which is Lebesgue integrable but not Riemann integrable.
   (c) A sequence f_n of functions in $L^1([0, 1])$ which converges pointwise to zero but satisfies $\int_0^1 f_n dm = 1$.

2. State Fatou's Lemma and the Dominated Convergence Theorem. Prove them using the Monotone Convergence Theorem.

3. Suppose $f \in L^1(\mu)$. Prove: for every $\varepsilon > 0$ there is a $\delta > 0$ such that $|\int_E f d\mu| < \varepsilon$ whenever $\mu(E) < \delta$.

4. Let f be absolutely continuous in $[\delta, 1]$ for each $\delta > 0$ and continuous and of bounded variation on $[0, 1]$. Prove that f is absolutely continuous on $[0, 1]$.

5. Evaluate
$$\lim_{n\to\infty} \int_0^1 \frac{x^2}{x^2 + (1 - nx)^2} dx.$$

6. Find a function f on $(0, \infty)$ such that $f \in L^p((0, \infty))$ if and only if $1 < p < 2$.

7. Show that the product of two absolutely continuous functions on the interval $[a, b]$ is absolutely continuous. Use this to derive a theorem about integration by parts.

8. (a) If $g \in L^1(\mathbb{R})$ show that there is a bounded measurable function f on $\mathbb{R}$ such that $\|f\|_\infty > 0$ and
$$\int_{\mathbb{R}} fg dm = \|g\|_1 \|f\|_\infty.$$
(b) If $g \in L^\infty(\mathbb{R})$ show that for each $\varepsilon > 0$ there is an integrable function f on $\mathbb{R}$ such that
$$\int_{\mathbb{R}} fg dm \geq (\|g\|_\infty - \varepsilon)\|f\|_1.$$

9. Suppose $1 \leq p \leq q \leq \infty$. Prove or disprove:
(a) $L^p([0, 1]) \subset L^q([0, 1])$.
(b) $\ell^p \subset \ell^q$.

10. Suppose $f \in L^1(\mathbb{R})$ and define
$$g(x) = \int_{\mathbb{R}} f(y) \exp(-(x - y)^2) dy.$$
Show that $g \in L^p(\mathbb{R})$ for every $p \in [1, \infty]$ and estimate $\|g\|_p$ in terms of $\|f\|_1$. (Note that $\int_{\mathbb{R}} \exp(-x^2) dx = \sqrt{\pi}$.)