The University of Alabama System

Joint Ph.D Program in Applied Mathematics

Linear Algebra and Numerical Linear Algebra JP Exam

May 2023

Instructions:

- This is a closed book examination. Once the exam begins, you have three and one half hours to do your best. You are required to do **seven of the eight problems for full credit.**
- Each problem is worth 10 points; parts of problems have equal value unless otherwise specified.
- Justify your solutions: cite theorems that you use, provide counter examples for disproof, give explanations, and show calculations for numerical problems.
- Begin each solution on a new page and write the last four digits of your university **student ID number**, and problem number, on every page. Please write only on one side of each sheet of paper.
- The use of calculators or other electronic gadgets is not permitted during the exam.
- Write legibly using dark pencil or pen.

1. Let T be a linear operator on a four dimensional complex vector space that satisfies the polynomial equation $P(T) = T^4 + 2T^3 - 2T - I = 0$, where I is the identity operator on V. Suppose that $|\text{trace}(T)| = 2$ and that $\dim \text{range}(T + I) = 2$. Give a Jordan canonical form of T

2. Let V be a vector space over a field $\mathbb{F}$. Suppose $T \in \mathcal{L}(V)$ has minimal polynomial $p(z) = 3 + 2z - z^2 + 5z^3 + z^4$.
(a) (2.5 pts) Prove that T is invertible.
(b) (7.5 pts) Find the minimal polynomial of T^{-1}

3. Suppose A is a normal matrix such that $A^5 = A^4$.
(a) (5 pts) Prove that A is self-adjoint.
(b) (3.5 pts) Give a counterexample to Part (a) if A is not normal.
(c) (1.5 pts) Prove or disprove that A is a projection matrix. (Recall that a matrix A is a projection matrix if $A^2 = A$.)

4. Let $A = \begin{bmatrix} 3 & -3 \\ 0 & 4 \\ 4 & -1 \end{bmatrix}$.

   (a) Find the QR factorization of A by Householder reflectors.

   (b) Use the results in (a) to find the least squares solution of $A\mathbf{x} = \mathbf{b}$, where $\mathbf{b} = [16 \;\; 11 \;\; 17]^T$.

5. Let V be a vector space of dimension n over a field F. For any nilpotent operator T on V, define the smallest integer p such that $T^p = 0$ as the index of nilpotency of T.

   (a) Suppose that N is nilpotent of index p. If $v \in V$ is such that $N^{p-1}(v) \neq 0$, prove that
   $$\{v, N(v), \ldots, N^{p-1}(v)\}$$
   is linearly independent.

   (b) Show that N is nilpotent of index n if and only if there is an ordered basis $v_1, v_2, \ldots, v_n$ of V such that the matrix of N with respect to the basis is of the form

$$\begin{bmatrix} 0 & 0 & 0 & \cdots & 0 & 0 \\ 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 \end{bmatrix}$$

(c) Show that an $n \times n$ matrix M over F is such that $M^n = 0$ and $M^{n-1} \neq 0$ if and only if M is similar to a matrix of the above form.

6. Prove that the largest singular value of a linear transformation $A \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^m)$ is equal to

$$\max_{x \in \mathbb{R}^n, y \in \mathbb{R}^m} \frac{< y, Ax >}{\|x\| \|y\|}.$$

7. Let N be a real $n \times n$ matrix of rank $n - m$ and nullity m. Let L be an $m \times n$ matrix whose rows form a basis of the left null space of N , and let R be an $n \times m$ matrix whose columns form a basis of the right null space of N. Put $Z = L^T R^T$. Finally, put $M = N + Z$.

(a) (1 points) For $x \in \mathbb{R}^n$, show that $N^T x = 0$ if and only if $x = L^T y$ for some $y \in \mathbb{R}^m$.

(b) (1 points) For $x \in \mathbb{R}^n$, show that $Nx = 0$ if and only if $x = Ry$ for some $y \in \mathbb{R}^m$.

(c) (2 points) Show that Z is an $n \times n$ matrix with rank m for which $N^T Z = 0$, $NZ^T = 0$ and $MM^T = NN^T + ZZ^T$.

(d) (6 points) Show that the eigenvalues of MM^T are precisely the positive eigen- values of NN^T and the positive eigenvalues of ZZ^T , and conclude that MM^T is nonsingular.

8. Let $F = \mathbb{C}$ and suppose that $T \in \mathcal{L}(V)$.

(a) Prove that the dimension of $\text{Im}(T)$ equals the number of nonzero singular values of T.

(b) Suppose that $\in \mathcal{L}(V)$ is positive semidefinite. Prove that T is invertible if and only if $< T(x), x >> 0$ for every $x \in V$ with $x \neq 0$.