University of Alabama System

Joint Ph.D. Program in Applied Mathematics

Joint Program Exam: Linear Algebra and Numerical Linear Algebra

May 5, 2022

- This is a closed book exam. The duration of the exam is **three and an half hours**.
- You are required to do **7 out of the 8 problems** for full credit.
- Each problem is worth 10 points; multiple parts of a given problem have equal weights (unless otherwise specified).
- You must justify your solutions: cite theorems that you use, provide counter examples for disproving theorems, give explanations and show all the calculations for the numerical problems.
- Start each solution on a new page. Write the last four digits of your university **student ID number** and the problem number on every page (do not put your name).
  Write only on one side of the page.
- No calculators are allowed. No other electronic devices are allowed.
- Please write legibly with a pen or a dark pencil.

1. (a) Show that for all $x \in \mathbb{R}^n$

$$||x||_2 \leq ||x||_1 \leq \sqrt{n}||x||_2. \tag{0.1}$$

(b) Make systematic use of the inequalities from Part (a) to prove that for all $A \in \mathbb{R}^{n\times n}$

$$||A||_1 \leq \sqrt{n}||A||_2 \leq n||A||_1. \tag{0.2}$$

2. Let $S \in \mathbb{C}^{n\times n}$ be skew-Hermitian, i.e., $S^* = -S$. Show the following:

(a) The eigenvalues of S are purely imaginary.

(b) The matrix $I - S$ is invertible.

(c) The matrix $U = (I - S)^{-1}(I + S)$ is unitary.

3. Let $A = U\Sigma V^T$ be the SVD of $A \in \mathbb{R}^{mxn}$ with nonzero singular values: $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_r > 0$. Prove the following:
(a) The rank(A) is r.
(b) $\|A\|_2 = \sigma_1$ where $\|A\|_2$ is the two-norm of A.
(c) $\|A\|_F \leq \sqrt{rank(A)}\|A\|_2$ where $\|\cdot\|_F$ is the Frobenius norm of A.

4. Let $A \in \mathbb{C}^{m\times n}$ with $m \geq n$.

(a) Given a vector $x \in \mathbb{C}^n$. Construct a Householder reflector with a unit vector v

$$P = I - 2v^*v$$

such that $Px = \|x\|y$. In addition, if $y = \pm e_1$ where $e_1^T = [1\ 0\ \cdots\ 0]$, then $Px = \pm\|x\|e_1$.

(b) Show that $A \in \mathbb{C}^{m\times n}$ has a QR decomposition with a unitary matrix Q that is a product of Householder reflectors.

5. Let A be an $n \times n$ complex matrix. Define $H = \frac{1}{2}(A + A^*)$ and $S = \frac{1}{2}(A - A^*)$. Prove that A is normal if every eigenvector of H is also an eigenvector of S.

6. Define $T \in \mathcal{L}(\mathbb{F}^n)$ by $T : (w_1, w_2, w_3, w_4)^T \to (0, w_2 + w_4, w_3, w_4)^T$.
(a) Determine the minimal polynomial of T .
(b) Determine the characteristic polynomial of T .
(c) Determine the Jordan form of T.

7. Let A and B be $n \times n$ Hermitian matrices over $\mathbb{C}$.
(a) If A is positive definite, show that there exists an invertible matrix P such that $P^*AP = I$ and P^*BP is diagonal.
(b) If A is positive definite and B is positive semidefinite, show that $det(A + B) \geq det(A)$.

8. Let $A \in \mathbb{C}^{n\times n}$ be a diagonalizable matrix so that

$$X^{-1}AX = D = \text{diag}(\lambda_1, \cdots, \lambda_n).$$

(a) Consider the perturbed matrix $A + \Delta A$. Show that if $D - \mu I$ is singular, then μ is an eigenvalue of A and

$$\min_{1\leq i\leq n} |\mu - \lambda_i| \leq \kappa_p(X)\|\Delta A\|_p$$

where $\|\cdot\|_p$ stands for any p-norm $(1 \leq p \leq \infty)$ and κ_p is the p-norm condition number.

(b) Let B be an arbitrary square matrix. If $\|B\| < 1$, then $I - B$ is invertible where

$$(I - B)^{-1} = I + B + B^2 + \cdots .$$

(c) Show that if μ is an eigenvalue of a perturbed matrix $A + \Delta A$, then

$$\min_{1 \le i \le n} |\mu - \lambda_i| \le \kappa_p(X) \|\Delta A\|_p.$$

(Use part b)