The University of Alabama System

Joint Ph.D Program in Applied Mathematics

Linear Algebra and Numerical Linear Algebra JP Exam

September 2023

Instructions:

- This is a closed book examination. Once the exam begins, you have three and one half hours to do your best. You are required to do **seven of the eight problems for full credit.**
- Each problem is worth 10 points; parts of problems have equal value unless otherwise specified.
- Justify your solutions: cite theorems that you use, provide counter examples for disproof, give explanations, and show calculations for numerical problems.
- Begin each solution on a new page and write the last four digits of your university **student ID number**, and problem number, on every page. Please write only on one side of each sheet of paper.
- The use of calculators or other electronic gadgets is not permitted during the exam.
- Write legibly using dark pencil or pen.

1. We consider the inner product space $\mathbb{R}^n$ with its standard inner product. $(< u, v >= u_1v_1 + ... + u_nv_n.)$ Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be defined by

$$T(z_1, z_2, ..., z_n) = (z_2 - z_1, z_3 - z_2, ..., z_1 - z_n).$$

   (a) Give an explicit expression for the adjoint, T^*.

   (b) Is T invertible? Explain.

   (c) Find the eigenvalues of T.

2. (a) Let $n \geq 2$ and Let V be an $n-$ dimensional vector space over $\mathbb{C}$ with a set of basis vectors $e_1, ..., e_n$. Let T be the linear map of V satisfying $T(e_i) = e_{i+1}, i = 1, ..., n-1$ and $T(e_n) = e_1$ Is T diagonalizable?

   (b) Let V be a finite-dimensional vector space and $T : V \to V$ a diagonalizable linear transformation. Let $W \subset V$ be a subspace which is mapped into itself by T. Show that the restriction of T to W is diagonalizable.

3. Let V be an n-dimensional inner product space over F.

   (a) Suppose $T \in L(V)$ and U is a subspace of V . Prove or disprove: $U^\perp$ is invariant under T^* if U is invariant under T.

   (b) Let T_1 and T_2 be two self-adjoint operators on V. Prove or disprove: $T_1T_2 + T_2T_1$ is also self-adjoint.

   (c) Let T be a self-adjoint operator on V . Show that T is a nonnegative self-adjoint operator on V if and only if the eigenvalues of T are all nonnegative real numbers

4. (a) Let $x := [1, 7, 2, 3, -1]^T$ and $y := [-4, 4, 4, 0, -4]^T$. Is there an orthogonal matrix Q such that $Qx = y$? If so, use exact arithmetic to find it. If not, explain why.

   (b) Consider a least squares problem

$$Ax = b, \quad A = \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}, x = \begin{bmatrix} x_1 \end{bmatrix}, b = \begin{bmatrix} 5 \\ -1 \\ 0 \end{bmatrix}. \tag{1}$$

   Compute a QR decomposition of the matrix A, with exact arithmetic, by using the Householder reflector method.

   (c) Compute the least squares solution of Eq. (1) based on the QR decomposition of the Part (b).

5. A matrix $A \in \mathbb{C}^{n\times n}$ is said to be normal, if $A^*A = AA^*$. Show the following:

   (a) If a normal matrix is triangular, then it is a diagonal matrix.

   (b) A matrix is normal if and only if it is unitarily similar to a diagonal matrix.

6. (a) Let A and B be normal matrices such that $ImA \perp ImB$. Prove that $A+B$ is a normal matrix.

   (b) Let P_1 and P_2 be projections. Prove that $P_1 + P_2$ is a projection if and only if $P_1P_2 = P_2P_1 = 0$.

7. Consider the 3 vectors

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ \epsilon \\ 0 \\ 0 \end{pmatrix}, \quad \mathbf{v}_2 = \begin{pmatrix} 1 \\ 0 \\ \epsilon \\ 0 \end{pmatrix}, \quad \mathbf{v}_3 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ \epsilon \end{pmatrix}.$$

   where $\epsilon \ll 1$.

   (a) Use the Classical Gram-Schmidt method to compute 3 **orthonormal** vectors $\mathbf{q}_1$, $\mathbf{q}_2$ and $\mathbf{q}_3$, making the approximation that $1 + \epsilon^2 \approx 1$ (that is replace any term containing ϵ^2 or smaller with zero, **but** retain terms containing ϵ). Are all the $\mathbf{q}_i$ $(i = 1, 2, 3)$ pairwise orthogonal? If not, why not?

   (b) Repeat the previous step using the modified Gram-Schmidt orthogonalization process. Are the $\mathbf{q}_i$ $(i = 1, 2, 3)$ pairwise orthogonal? If not, why not?

8. (a) Let

$$M = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & -1 \end{pmatrix}.$$

   Find a matrix T such that $T^{-1}MT$ is diagonal, or prove that such a matrix does not exist.

   (b) Find a matrix whose minimal polynomial is $x^2(x-1)^2$, whose characteristic polynomial is $x^4(x-1)^3$ and whose rank is 4.