University of Alabama System

Joint Ph.D. Program in Applied Mathematics

Joint Program Exam: Linear Algebra and Numerical Linear Algebra

September, 2020

- This is a closed book exam. The duration of the exam is **three and an half hours**.
- You are required to do **7 out of the 8 problems** for full credit.
- Each problem is worth 10 points; multiple parts of a given problem have equal weights (unless otherwise specified).
- You must justify your solutions: cite theorems that you use, provide counter examples for disproving theorems, give explanations and show all the calculations for the numerical problems.
- Start each solution on a new page. Write the last four digits of your university **student ID number** and the problem number on every page (do not put your name). Write only on one side of the page.
- No calculators are allowed. No other electronic devices are allowed.
- Please write legibly with a pen or a dark pencil.

1. (a) Let $T : V \to V$ be a linear operator. Suppose $v_1, v_2, \dots, v_n$ are non-zero vectors in V such that $T(v_1) = 0$ and $T(v_i) = v_{i-1}$ for $2 \leq i \leq n$. Prove that $\{v_1, ..., v_n\}$ is a linearly independent set.

   (b) Let $B = \{u_1, u_2, \dots, u_n\}$ be a basis of a vector space V. Let $C = \{v_1, v_2, \dots, v_m\}$ be a linearly independent set in V. Prove that there is an integer k, $1 \leq k \leq n$, such that the vectors $u_k, v_2, \dots, v_m$ are linearly independent.

2. Let V be an n-dimensional, complex inner product space with inner product $\langle \cdot, \cdot \rangle$. Let $S \subset V$ be a subset in V and $S^\perp = \{v \in V \mid \langle v, s \rangle = 0 \text{ for all } s \in S\}$.

   (a) Show that $S^\perp$ is a subspace of V.

   (b) Suppose that $W \subset V$ is a subspace of V with an orthonormal basis $\{b_1, \dots, b_m\}$, and $\{c_1, \dots, c_l\}$ is an orthonormal basis for $W^\perp$. Prove that $\{b_1, ..., b_m, c_1, ..., c_l\}$ is an orthonormal basis for V and that $l = n - m$.

3. Let T be a linear operator on $\mathbb{C}^4$ defined by

$$T : (w_1, w_2, w_3, w_4)^T \mapsto (0, w_2 - w_4, w_3, w_4)^T, \quad \forall (w_1, w_2, w_3, w_4)^T \in \mathbb{C}^4.$$

   (a) Determine the minimal polynomial of T.

   (b) Determine the characteristic polynomial of T.

   (c) Determine the Jordan form of T.

4. For a matrix $A \in \mathbb{C}^{m \times n}$ we define its adjoint matrix $A^* \in \mathbb{C}^{n \times m}$ as $A^* = \overline{A^T} = \bar{A}^T$.

   (a) Prove that A and A^*A have the same null space.

   (b) Use (a) to show that A^*A is nonsingular if and only if the rank of A is n.

5. Consider the system

$$\begin{pmatrix} \varepsilon & 1 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

   Assume that $|\varepsilon| \ll 1$. Solve the system by using the LU decomposition with and without partial pivoting and adopting the following rounding off models (at all stages of the computation!):

$$a + b\varepsilon = a$$

   (for $a \neq 0$) and

$$a + b/\varepsilon = b/\varepsilon$$

   (for $b \neq 0$). Also find the exact solution, compare, and make comments.

6. Let $Q \in \mathbb{R}^{n\times n}$ be an orthogonal matrix (i.e. $Q^TQ = QQ^T = I$).

   (a) For any $x \in \mathbb{R}^n$, show that $\|Qx\|_2 = \|x\|_2$ and hence $\|Q\|_2 = 1$.

   (b) For any $A \in \mathbb{R}^{n\times n}$, show that $\|QA\|_2 = \|A\|_2$.

   (c) For any $A \in \mathbb{R}^{n\times n}$, define $B = Q^{-1}AQ$ (i.e. A and B are orthogonally similar), show that $\|B\|_2 = \|A\|_2$.

7. Let $A \in \mathbb{R}^{n\times m}$, $n > m$ and $\text{rank}(A) = m$. The singular value decomposition (SVD) of A is $A = U\Sigma V^T$, where $U \in \mathbb{R}^{n\times n}$ and $V \in \mathbb{R}^{m\times m}$ are orthogonal, and $\Sigma \in \mathbb{R}^{n\times m}$ has singular values $\sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_m > 0$.

   (a) Determine the SVD decompositions of the matrices

   $$(A^TA)^{-1}, \quad (A^TA)^{-1}A^T, \quad A(A^TA)^{-1}, \quad \text{and} \quad A(A^TA)^{-1}A^T$$

   in terms of the SVD of A. Please specify the dimensions and elements of the obtained Σ matrices.

   (b) Use the results of part (a) to determine the matrix 2−norms

   $$\|(A^TA)^{-1}\|_2, \quad \|(A^TA)^{-1}A^T\|_2, \quad \|A(A^TA)^{-1}\|_2, \quad \text{and} \quad \|A(A^TA)^{-1}A^T\|_2.$$

8. Consider a least squares problem

   $$\begin{bmatrix} 2 & 3 \\ 2 & 4 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 7 \\ 3 \\ 1 \end{bmatrix}.$$

   (a) Compute a QR decomposition of the matrix, with exact arithmetic, by using the Householder reflector method.

   (b) Compute the Least squares solution based on the QR decomposition of part (a).