University of Alabama System

Joint Ph.D. Program in Applied Mathematics

Joint Program Exam: Linear Algebra and Numerical Linear Algebra

May, 2020

- This is a closed book exam. The duration of the exam is **three and an half hours**.
- You are required to do **7 out of the 8 problems** for full credit.
- Each problem is worth 10 points; multiple parts of a given problem have equal weights (unless otherwise specified).
- You must justify your solutions: cite theorems that you use, provide counter examples for disproving theorems, give explanations and show all the calculations for the numerical problems.
- Start each solution on a new page. Write the last four digits of your university **student ID number** and the problem number on every page (do not put your name). Write only on one side of the page.
- No calculators are allowed. No other electronic devices are allowed.
- Please write legibly with a pen or a dark pencil.

1. Let V be a real finite dimensional inner product space and let $T : V \to V$ be a linear operator. Assume that $< Tv, w >=< v, Tw >$ for all $v, w \in V$.

   (a) Prove that if λ and μ are distinct eigenvalues of T then the corresponding eigenspaces V_λ and V_μ are orthogonal.

   (b) If W is a subspace of V, prove that $T(W) \subseteq W$ implies that $T(W^\perp) \subseteq W^\perp$.

   (c) Prove that there exists an eigenvector $v_1 \in V$ for T in V with associated (real) eigenvalue λ_1. Do not use a big theorem; prove directly. You may assume the fundamental theorem of algebra however.

   (d) Prove that there exists an orthonormal basis of V consisting of eigenvectors for T.

2. Let $n > 0$ be an integer. Find all $n \times n$ matrices A with complex entries such that A is Hermitian ($A = A^*$) and

$$A^3 = 2A + 4I\,.$$

3. Let $A = (a_{ij}) \in \mathbb{R}^{n\times n}$ be symmetric and positive definite. Show that

$$\det \begin{pmatrix} a_{11} & \cdots & a_{1n} & x_1 \\ & \cdots & & \vdots \\ a_{n1} & \cdots & a_{nn} & x_n \\ x_1 & \cdots & x_n & 0 \end{pmatrix} < 0$$

for every nonzero vector $x = (x_1, \ldots, x_n)^t \in \mathbb{R}^n$.

4. Let A and B be $m \times n$ and $n \times p$ matrices over $\mathbb{R}$, respectively.

   (a) Prove that $\dim(Null(AB)) \leq \dim(Null(A)) + \dim(Null(B))$.
   (Hint: it may be convenient to let $V = \{x \in \mathbb{R}^p \,:\, ABx = 0\}$ and $W = \{y = Bx \,:\, x \in \mathbb{R}^p\,,\, Ay = 0\}$. Then consider the map $T_B : V \to W$ defined by $T_B(x) = Bx$ for all $x \in V$).

   (b) Prove that $rank(A) + rank(B) \leq rank(AB) + n$.

5. Let $v, x, y \in \mathbb{C}^n$ be nonzero vectors.

   (a) Prove that $U = I - vv^*$ is unitary if and only if $\|v\|_2 = \sqrt{2}$.

   (b) Prove that if $\|x\|_2 = \|y\|_2$ and if the inner product of x and y is real, then there exists a unitary matrix U of the form $I - vv^*$ such that $Ux = y$ for some vector v.

(c) Exploit the above results to find a QR factorization of the matrix A given below, such that $A = QR$ where Q is unitary and R is an upper triangular matrix.

$$A = \begin{pmatrix} 4 & 4 & 1 \\ 3 & -2 & 7 \\ 0 & 3 & 1 \end{pmatrix}.$$

6. Let $U \in \mathbb{R}^{m\times r}$, $m \geq r$, be a matrix with orthonormal columns (i.e. $U^tU = I$).

   (a) For any $x \in \mathbb{R}^r$, show that $\|Ux\|_2 = \|x\|_2$ (and hence $\|U\|_2 = 1$).

   (b) For any $A \in \mathbb{R}^{r\times n}$, show that $\|UA\|_2 = \|A\|_2$. Also for any $B \in \mathbb{R}^{s\times m}$, and $r = m$ (i.e. U is a square matrix), show that $\|BU\|_2 = \|B\|_2$. What happens if U is not a square matrix?

7. A matrix A is normal if $AA^* = A^*A$, where A^* is the conjugate transpose of A. Prove that

   (a) if A is a normal matrix then A and A^* have same eigenvectors.

   (b) if A is a normal matrix and two vectors x and y are eigenvectors of A corresponding to different eigenvalues, then the vectors x and y are orthogonal.

   (c) If A is a normal and upper triangular matrix then A is diagonal.

8. Let $A \in \mathbb{R}^{m\times n}$ and
$$A = U\begin{pmatrix} \Sigma & 0 \\ 0 & 0 \end{pmatrix}V^t$$
be a singular value decomposition (SVD) of A, where $U = (u_1\ u_2\ \cdots\ u_m) \in \mathbb{R}^{m\times m}$, $V = (v_1\ v_2\ \cdots\ v_n) \in \mathbb{R}^{n\times n}$ are orthogonal matrices, and $\Sigma = \text{diag}\,(\sigma_1\ \sigma_2\ \ldots\ \sigma_p)$, with $\sigma_1 \geq \ldots \geq \sigma_p \geq 0, 1 \leq p \leq \min(m, n)$. Prove that

   (a) $Av_i = \sigma_i u_i \quad 1 \leq i \leq p$,

   (b) $\|A\|_2 = \sigma_1$,

   (c) if we define r by $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_r > \sigma_{r+1} = \cdots = \sigma_p = 0$, then rank($A$)=$r$, the null space $\mathcal{N}(A) = \text{span}\{v_{r+1}, \ldots, v_n\}$, and the range $\mathcal{R}(A) = \text{span}\{u_1, \ldots, u_r\}$.