The University of Alabama System

Joint Ph.D Program in Applied Mathematics

Linear Algebra and Numerical Linear Algebra JP Exam

September 2018

Instructions:

- This is a closed book examination. Once the exam begins, you have three and one half hours to do your best. You are required to do **seven of the eight problems for full credit.**
- Each problem is worth 10 points; parts of problems have equal value unless otherwise specified.
- Justify your solutions: cite theorems that you use, provide counter examples for disproof, give explanations, and show calculations for numerical problems.
- Begin each solution on a new page and write the last four digits of your university **student ID number**, and problem number, on every page. Please write only on one side of each sheet of paper.
- The use of calculators or other electronic gadgets is not permitted during the exam.
- Write legibly using dark pencil or pen.

1. Let $A \in \mathbb{R}^{3\times 3}$ be an unknown matrix, and let

$$\mathbf{v}_1 := \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \mathbf{v}_2 := \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \mathbf{v}_3 := \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \in \mathbb{R}^3 .$$

   Further let $S := [\mathbf{v}_1|\mathbf{v}_2|\mathbf{v}_3]$ be the real 3×3 matrix with columns $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$. Finally, let

$$\ker(A + 2I_3) = \text{span} < \mathbf{v}_1 > \quad \text{and} \quad \ker(A - I_3) = \text{span} < \mathbf{v}_2, \mathbf{v}_3 > ,$$

   where I_3 is the 3×3 identity matrix.

   (a) Prove that A is diagonalizable and give the characteristic polynomial c_A in factored form. Further, give all eigenvalues of A with their geometric and algebraic multiplicities. Finally, give the minimal polynomial m_A of A.

   (b) Compute the matrix A.

2. Let D be in $\mathbb{R}^{n\times n}$ and diagonal with entries $d_1 < d_2 < \ldots < d_n$. Let Z be a symmetric rank 1 matrix with non-zero eigenvalue ρ and no zero entries. Prove that if λ is an eigenvalue of $D + Z$ and $\mathbf{v}$ is a corresponding eigenvector, then

   (a) $Z\mathbf{v} \neq 0$.

   (b) D and $D + Z$ do not have any common eigenvalues.

3. Let $A, B, C, D \in \mathbb{F}^{n\times n}$ be square matrices, where the filed $\mathbb{F}$ is either real or complex field.

   (a) Prove that $T(X) = AXB + CX + XD$ is a linear transformation on $\mathbb{F}^{n\times n}$.

   (b) If $C = D = 0$, prove that T is invertible if and only if A and B are invertible.

   (c) Let $A = B = C = D$, and equip $\mathbb{F}^{n\times n}$ with the matrix $2-$norm. Prove a non-trivial upper bound on the operator norm of T in terms of the singular values of A. You do not have to prove that $\|A\|_2$ is a submultiplicative norm.

4. Prove that the largest singular value of a linear transformation $A \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^m)$ is equal to

$$\max_{\mathbf{x}\in\mathbb{R}^n, \mathbf{y}\in\mathbb{R}^m} \frac{< \mathbf{y}, A\mathbf{x} >}{\|\mathbf{x}\|\|\mathbf{y}\|} .$$

5. Let $A \in \mathbb{R}^{m\times n}$ and $b \in \mathbb{R}^m$.

   (a) Suppose that $m > n$ and $\text{rank}(A) = n$. Show that the solution of

   $$\min_{x\in\mathbb{R}^n} \|Ax - b\|_2$$

   is equal to $\bar{x} = (A^T A)^{-1} A^T b$.

   (b) Suppose, on the other hand, that $m < n$ and $\text{rank}(A) = m$. Give a formula for the minimum ℓ_2-norm solution $\hat{x}$ of $Ax = b$. Show that all solutions of the linear system $Ax = b$ have the form $\hat{x} + d$, where d is an element of a particular $(n - m)$-dimensional subspace of $\mathbb{R}^n$. What is this subspace?

6. (a) Suppose that $A \in \mathbb{R}^{m\times m}$ is symmetric and nonsingular with LU factorization $A = LU$. Show that there exists a unique diagonal matrix $D \in \mathbb{R}^{m\times m}$ such that

   $$A = LDL^T.$$

   (b) Show that $A \in \mathbb{R}^{m\times m}$ is *symmetric positive definite* if and only if A has a *Cholesky factorization*:

   $$A = BB^T,$$

   where B is a lower triangular matrix with positive diagonal entries.

7. (a) Prove that if $\kappa(A) := \|A\|\|A^{-1}\|$ is defined by any matrix norm (induced by a vector norm), then $\kappa(A) \le \kappa(A)\kappa(B)$ for any $n\times n$ invertible matrices.

   (b) Compute the condition numbers $\kappa_1(A), \kappa_2(A)$ and $\kappa_\infty(A)$ for

   $$A = \begin{bmatrix} 1 & 1 - \frac{1}{n} \\ 1 + \frac{1}{n} & 1 \end{bmatrix}$$

   where $n \ge 2$.

8. Let V be a finite dimensional vector space with inner product $\langle \cdot, \cdot \rangle$ and let T be a self-adjoint operator on V. Prove that there exists a self-adjoint operator S on V such that $T = S^2$ if and only if $\langle Tx, x\rangle \ge 0$ for all $x \in V$.