University of Alabama System

Joint Ph.D. Program in Applied Mathematics

Joint Program Exam: Linear Algebra and Numerical Linear Algebra

May 2018

- This is a closed book exam. The duration of the exam is **three and an half hours**.
- You are required to do **7 out of the 8 problems** for full credit.
- Each problem is worth 10 points; multiple parts of a given problem have equal weights (unless otherwise specified).
- You must justify your solutions: cite theorems that you use, provide counter examples for disproving theorems, give explanations and show all the calculations for the numerical problems.
- Start each solution on a new page. Write the last four digits of your university **student ID number** and the problem number on every page (do not put your name).
  Write only on one side of the page.
- No calculators are allowed. No other electronic devices are allowed.
- Please write legibly with a pen or a dark pencil.

1. True or False? For each of the following statements prove the truth or demonstrate the falsity by a counter-example, where $A \in \mathbb{R}^{n\times n}$, $x \in \mathbb{R}^n$.

   (a) $x^T Ax = x^T(\frac{A+A^T}{2})x$.

   (b) $A^k = 0$ for some positive integer k implies $A = 0$.

   (c) If A is orthogonal then $Ax = b$ can be solved in $O(n^2)$ flops.

   (d) If A is symmetric positive definite, and X is nonsingular, then $X^T AX$ is symmetric positive definite.

   (e) Let $A = I$, $\|\cdot\|_F^2 = \mathrm{tr}(A^*A)$ and $\|\cdot\|_2$ is the matrix 2-norm, $\|I\|_2 = \|I\|_F = 1$.

2. (a) Compute the condition numbers with respect to the matrix 1-norm and ∞-norm, $\kappa_1(A)$ and $\kappa_\infty(A)$, for the matrix

$$A = \begin{bmatrix} 3 & 2\sqrt{2} \\ 2\sqrt{2} & 3 \end{bmatrix}.$$

   (b) Show that for every nonsingular 2×2 matrix, we have $\kappa_1(A) = \kappa_\infty(A)$.

3. Suppose that $A \in \mathbb{R}^{m\times n}$. Let $A = U\Sigma V^T$ be the SVD of A, $U \in \mathbb{R}^{m\times n}$, $V \in \mathbb{R}^{n\times n}$, $\Sigma = \mathrm{diag}(\sigma_1, ..., \sigma_n)$.

   (a) Determine the matrix 2-norm of A, $\|A\|_2$, using the SVD of A.

   (b) Determine an eigendecomposition of $A^T A$ in terms of the SVD of A.

   (c) Determine an eigendecomposition of AA^T in terms of the SVD of A.

   (d) Let $A = \begin{bmatrix} 3 \\ 0 \\ 4 \end{bmatrix}$

      (i) Find the singular values and singular vectors of A. (Hint: use (b) and (c)).

      (ii) Express A as the SVD form $A = U\Sigma V^T$, where U and V are (square) orthogonal matrices and Σ is a "diagonal" matrix.

      (iii) Solve the normal equations by an appropriate method. Give reasons for your choice of the method.

4. Let $A_n(c)$ be the $n \times n$ matrix defined by

$$A_n(c) = \begin{bmatrix} 1 & 1 & \dots & 1 \\ \vdots & \vdots & & \vdots \\ 1 & 1 & \dots & 1 \\ c & c & c & c \end{bmatrix}, \text{ (all entries 1 exept the entries of the last row are equal to } c).$$

   For every integer $n \geq 2$ and $c \in \mathbb{C}$, determine a Jordan canonical form $J_n(c)$ that is similar to $A_n(c)$.

5. (a) Use the Schur factorization to show that if $\boldsymbol{A} \in \mathbf{C}^{n\times n}$ is Hermitian then $\boldsymbol{A}$ is unitarily diagonalizable.

   (b) Now suppose that $\boldsymbol{A} \in \mathbf{C}^{m\times m}$ is Hermitian. Show that $\boldsymbol{A}$ is positive definite if and only if every eigenvalue of $\boldsymbol{A}$ is strictly positive.

6. Let $Ax \approx b$ be an overdetermined system where $A \in \mathbb{C}^{m\times n}$. Define $F(x) = \|b - Ax\|_2$.

   (a) A minimizer x^* of $F(x)$ is an exact solution of the system $Ax = \hat{b}$ where $\hat{b}$ is the orthogonal projection of b onto the Range A. (Hint: define $r = b - \hat{b}$)

   (b) Show that $(\text{Range } A)^{\perp} = \text{Ker } A^*$.

   (c) A minimizer x^* of $F(x)$ is a solution of the normal equation $A^*Ax = A^*b$. Use part b.

7. Suppose that $\boldsymbol{A} \in \mathbf{R}^{n\times n}$ has rank equal to $r \le n$. Suppose further that $\boldsymbol{A}$ and $\boldsymbol{A}^T$ have QR factorizations

$$\boldsymbol{A} = \boldsymbol{Q}\boldsymbol{R} \qquad \boldsymbol{A}^T = \tilde{\boldsymbol{Q}}\tilde{\boldsymbol{R}}.$$

   (a) Use the QR factorizations of $\boldsymbol{A}$ and $\boldsymbol{A}^T$ to provide orthonormal bases for the column space and row spaces of $\boldsymbol{A}$, and the null spaces of $\boldsymbol{A}$ and $\boldsymbol{A}^T$.

   (b) Use the result of Part (a) to establish the *rank-nullity* theorem:

$$n = \text{rank}(\boldsymbol{A}) + \dim(\text{Null}(\boldsymbol{A})) = r + \dim(\text{Null}(\boldsymbol{A})).$$

8. Let $A \in \mathbf{R}^{n\times n}$ be symmetric, with eigenvalues having distinct magnitudes:

$$|\lambda_1| > |\lambda_2| > |\lambda_3| > \cdots > |\lambda_n|.$$

   Let $\mathbf{v_i} \in \mathbf{R^n}$ denote the unit eigenvector of $\boldsymbol{A}$ corresponding to λ_i.

   (a) Suppose that the *power method* is applied to $\boldsymbol{A}$ to generate the sequence of iterates $\{\mathbf{q_i}\}_{\mathbf{i=0}}^{\infty}$, with initial iterate $\mathbf{q_0}$ satisfying $\mathbf{q_0^T v_1} \neq \mathbf{0}$. You may assume that the iterates are normalized each iteration, i.e. $\mathbf{q_{k+1}} = \boldsymbol{A}\mathbf{q_k}/\|\boldsymbol{A}\mathbf{q_k}\|_\mathbf{2}$.
   Show that this sequence converges to either $\mathbf{v_1}$ or $-\mathbf{v_1}$.

   (b) Now suppose that we apply the power method to $\boldsymbol{A}$ simultaneously for a set of n independent starting vectors, stored as $\boldsymbol{Q}_0$. That is consider the sequence of matrices $\{\boldsymbol{Q}_k\}_{k=1}^{\infty}$ given by

$$\boldsymbol{Q}_{k+1} = \boldsymbol{A}\boldsymbol{Q}_k,$$

   followed by normalization of the columns of $\boldsymbol{Q}_{k+1}$ so that each column has ℓ_2-norm equal to 1 as in Part (a). Suppose further that no column of $\boldsymbol{Q}_0$ is orthogonal to $\mathbf{v_1}$.
   What is the limit of the sequence $\{\boldsymbol{Q}_k\}_{k=0}^{\infty}$ as $k \to \infty$?

   (c) Consider the *QR iteration* and *simultaneous iteration* methods. QR iteration generates a sequence of iterates $\{\boldsymbol{A}_k\}_{k=0}^{\infty}$ starting with $\boldsymbol{A}_0 = \boldsymbol{A}$ by taking the QR factorization of the current iterate and then recombining to form the next iterate:

$$\boldsymbol{Q}\boldsymbol{R} = \boldsymbol{A}_k \qquad \boldsymbol{A}_{k+1} = \boldsymbol{R}\boldsymbol{Q}, \qquad k = 0, 1, 2, \ldots.$$

   Simultaneous iteration performs a power method iteration followed by reorthogonalization to generate the sequence of iterates $\{\tilde{\boldsymbol{Q}}_k\}_{k=0}^{\infty}$:

$$\boldsymbol{Z} = \boldsymbol{A}\tilde{\boldsymbol{Q}}_k \qquad \tilde{\boldsymbol{Q}}_{k+1}\tilde{\boldsymbol{R}}_{k+1} = \boldsymbol{Z}, \qquad k = 0, 1, 2, \ldots.$$

   Suppose that $\boldsymbol{Q}_0 = \boldsymbol{I}$ in simultaneous iteration. Show that the sequences generated by QR iteration and simultaneous iteration satisfy

$$\boldsymbol{A}_k = \tilde{\boldsymbol{Q}}_k^T \boldsymbol{A} \tilde{\boldsymbol{Q}}_k$$

   for all k.