The University of Alabama System

Joint Ph.D Program in Applied Mathematics

Linear Algebra and Numerical Linear Algebra JP Exam

September 2017

Instructions:

- This is a closed book examination. Once the exam begins, you have three and one half hours to do your best. You are required to do **seven of the eight problems for full credit.**
- Each problem is worth 10 points; parts of problems have equal value unless otherwise specified.
- Justify your solutions: cite theorems that you use, provide counter examples for disproof, give explanations, and show calculations for numerical problems.
- Begin each solution on a new page and write the last four digits of your university **student ID number**, and problem number, on every page. Please write only on one side of each sheet of paper.
- The use of calculators or other electronic gadgets is not permitted during the exam.
- Write legibly using dark pencil or pen.

1. Two $n \times n$ real matrices A and B are called *simultaneously diagonalizable* if there is an invertible matrix $S \in \mathbb{R}^{n\times n}$ such that $S^{-1}AS$ and $S^{-1}BS$ both are diagonal matrices. Let A and B be two $n \times n$ real matrices. Prove that

   (a) If A and B are simultaneously diagonalizable, then $AB = BA$.

   (b) If $AB = BA$ and if A has n different eigenvalues, then A and B are simultaneously diagonalizable.

2. Let A and B be two complex matrices, and suppose A and B have the same eigenvectors. Show that if the minimal polynomial of A is $(x+1)^2$ and the characteristic polynomial of B is x^5, then $B^3 = 0$.

3. Let A be a full column rank $n \times k$ matrix (so $k \leq n$) and $\mathbf{b}$ be a column vector of size n. We want to minimize the squared Euclidean norm $L(x) := \|A\mathbf{x} - \mathbf{b}\|_2^2$ with respect to $\mathbf{x}$.

   (a) Prove that, if rank(A)=k, then A^TA is invertible.

   (b) Directly derive the normal equations by minimizing $L(\mathbf{x})$, and then provide the closed-form expression for x that minimizes $L(\mathbf{x})$.

   (c) We consider a QR factorization of A where Q is $n \times k$ and R is $k \times k$ matrices. Show that an equivalent solution for $\mathbf{x}$ is $\mathbf{x} = R^{-1}Q^T\mathbf{b}$.

4. Consider the map $\phi : \mathbb{R}^4 \to \mathbb{R}^{2\times 2}$ where

$$\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} \mapsto \begin{bmatrix} a+b-c & c-d \\ 2a+c & a-b+d \end{bmatrix}$$

   (a) Is ϕ bijective? Prove your claim.

   (b) Compute $(a,b,c,d)^T$ such that $\phi((a,b,c,d)^T) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ or decide that this is not possible.

5. (a) Suppose $A \in \mathbb{C}^{m\times m}$ has an SVD $A = U\Sigma V^*$. Find an eigenvalue decomposition of the $2m \times 2m$ Hermitian matrix

$$\begin{bmatrix} 0 & A^* \\ A & 0 \end{bmatrix}.$$

(b) If A is Hermitian with eigenvalues $\lambda_1, \cdots, \lambda_n$, show that

$$\kappa_2(A) = \frac{\max_i |\lambda_i|}{\min_i |\lambda_i|}$$

where $\kappa_2(A)$ is the 2−norm condition number.

6. Suppose that $A \in \mathbb{C}^{n\times n}$ is normal, i.e., $AA^* = A^*A$. Show that if A is also upper triangular, it must be diagonal. Use this to show that A is normal if and only if A has n orthonormal eigenvectors.

7. Let A be an $n \times n$ real matrix of full rank with eigenvalues $\lambda_1, \lambda_2, \ldots.., \lambda_n$ and let X be a matrix that diagonalizes A, i.e. $X^{-1}AX = D$ where D is a diagonal matrix. If $A' = A + E$ and λ' is an eigenvalue of A', prove that

$$\min_{1\le i\le n} |\lambda' - \lambda_i| \le \kappa_2(X)\|E\|_2$$

where $\kappa_2(X)$ is the 2-norm condition number of X.

8. Let $A \in \mathbb{R}^{n\times n}$ be an invertible matrix, $\mathbf{b} \in \mathbb{R}^n$, and $\mathbf{x} \in \mathbb{R}^n$ denote the unique exact solution of the linear system $A\mathbf{x} = \mathbf{b}$. Let also

$$(A + \delta A)\mathbf{y} = \mathbf{b} + \delta b\,.$$

Assume that

$$\frac{\|\delta A\|}{\|A\|} \le \epsilon\,, \quad \frac{\|\delta \mathbf{b}\|}{\|\mathbf{b}\|} \le \epsilon\,,$$

and $\epsilon\kappa(A) = r < 1$, where $\kappa(A)$ is the condition numerber of A. Show that $A + \delta A$ is invertible and

$$\frac{\|\mathbf{y}\|}{\|\mathbf{x}\|} \le \frac{1+r}{1-r}\,.$$