University of Alabama System

Joint Ph.D. Program in Applied Mathematics

Joint Program Exam: Linear Algebra and Numerical Linear Algebra

May 2017

- This is a closed book exam. The duration of the exam is **three and an half hours**.
- You are required to do **7 out of the 8 problems** for full credit.
- Each problem is worth 10 points; multiple parts of a given problem have equal weights (unless otherwise specified).
- You must justify your solutions: cite theorems that you use, provide counter examples for disproving theorems, give explanations and show all the calculations for the numerical problems.
- Start each solution on a new page. Write the last four digits of your university **student ID number** and the problem number on every page (do not put your name). Write only on one side of the page.
- No calculators or other electronic devices are allowed.
- Please write legibly with a pen or a dark pencil.

1. Define $T : P_2(\mathbb{R}) \to P_3(\mathbb{R})$ by $T(f(x)) = xf(x) + f'(x)$.

   (a) Prove that T is a linear operator.

   (b) Find basis β for N(T).

   (c) Find basis γ for R(T).

   (d) Compute the nullity and rank of T and verify the dimension theorem.

   (e) Is T $1-$ to -1 or onto? Justify your answer.

2. Let $A, Q_0 \in \mathbb{R}^{m\times m}$. Define sequences of matrices Z_k, Q_k and R_k by

$$Z_k = AQ_{k-1}, \quad Q_kR_k = Z_k, \qquad k = 1, 2, \cdots,$$

   where Q_kR_k is an QR factorization of Z_k. Suppose $\lim_{k\to\infty} R_k = R_\infty$ exists.
   (a) Does it necessarily $\lim_{k\to\infty} Q_k = Q_\infty$ exist? Justify your answer.
   (b) Determine the eigenvalues of A in terms of R_∞ if $\lim_{k\to\infty} Q_k = Q_\infty$ exists.

3. Prove the following:

   (a) For every vector $z \in \mathbb{C}^n$, we have $\|z\| = \max_{\|y\|=1} |\langle y, z\rangle|$.

   (b) Use part a to prove $\|A\| = \|A^*\|$.

4. Let $u \in \mathbb{R}^n$ and let

$$P = I - \frac{2}{u^Tu}uu^T,$$

   a reflector matrix.

   (a) Show P is orthogonal.

   (b) Show that $P(x) = -v + w$ for $x = v + w$ where $v = \mu u$ and w is orthogonal to u.

5. Let $A \in \mathbb{R}^{n\times n}$ be a nonsingular matrix. Show that $\min\{\frac{\|\delta A\|_2}{\|A\|_2} \mid A+\delta A \text{ is singular}\} = 1/\kappa_2(A)$. (That is the relative distance to the nearest singular matrix is $1/\kappa_2(A)$.) Here $\kappa_2(A)$ is the 2-norm condition number of a matrix A defined to be $\kappa_2(A) = \|A\|_2\|A^{-1}\|_2$.

6. Given a matrix $A \in \mathbb{C}^{m\times n}$, if its compact SVD is $A = U\Sigma V^*$, then its pseudoinverse is $A^\dagger = V\Sigma^{-1}U^*$.

   (a) Prove that $x = A^\dagger b$ is one solution to the least squares problem $\min_x \|Ax - b\|_2^2$

   (b) Show that $x = A^\dagger b$ is the one with the smallest ℓ_2-norm among all solutions to $\min_x \|Ax-b\|_2^2$ [Hint: note that A may not be column full-rank, so the least squares problem can have infinitely many solutions].

7. Given a symmetric positive definite and tridiagonal matrix $A \in \mathbb{R}^{m\times m}$, let $A = QR$ be its full QR factorization, where Q is orthogonal, and R is upper-triangular. Prove that RQ is still symmetric positive definite and tridiagonal. Hence, the QR algorithm for eigenvalue problems maintains the symmetric tridiagonal form of the matrix. [Hint: explore the structure of R and Q]

8. Let v_1 and v_2 be two eigenvectors of an m by m matrix A.

(a) If they correspond to different eigenvalues λ_1 and λ_2, i.e., $\lambda_1 \neq \lambda_2$, then v_1 and v_2 are linearly independent.

(b) Use part a to show that if every eigenvalue of $A \in \mathbb{C}^{m \times m}$ has geometric multiplicity equal to its algebraic multiplicity, then A is diagonalizable, i.e., there is a nonsingular matrix V such that $V^{-1}AV$ is diagonal.