The University of Alabama System

Joint Ph.D Program in Applied Mathematics

Linear Algebra and Numerical Linear Algebra JP Exam

September 2016

Instructions:

- This is a closed book examination. Once the exam begins, you have three and one half hours to do your best. You are required to do **seven of the eight problems for full credit.**
- Each problem is worth 10 points; parts of problems have equal value unless otherwise specified.
- Justify your solutions: cite theorems that you use, provide counter examples for disproof, give explanations, and show calculations for numerical problems.
- Begin each solution on a new page and write the last four digits of your university **student ID number**, and problem number, on every page. Please write only on one side of each sheet of paper.
- The use of calculators or other electronic gadgets is not permitted during the exam.
- Write legibly using dark pencil or pen.

1. Let $a, b \in \mathbb{R}$ such that $a \neq b$. Let $A \in \mathbb{R}^{6\times 6}$ such that the characteristic polynomial of A is $C(x) := (x-a)^4(x-b)^2$ and the minimal polynomial of A is $m(x) := (x-a)^2(x-b)$. Describe all possible Jordan forms for A.

2. Let V be a finite dimensional real vector space. Let W_1 and W_2 be subspaces of V. We define the following operations:

$$(w_1, w_2) + (w_1', w_2') := (w_1 + w_1', w_2 + w_2')$$

and

$$\alpha * (w_1, w_2) := (\alpha w_1, \alpha w_2)$$

for all $(w_1, w_2) \in W_1 \times W_2$ and $(w_1', w_2') \in W_1 \times W_2$ and all $\alpha \in \mathbb{R}$. The set $W_1 \times W_2$ is a vector space with respect to these operations.

   (a) Let $U := \{(u, -u) : u \in W_1 \cap W_2\}$. Prove that U is a subspace of $W_1 \times W_2$. Also prove that U is isomorphic to $W_1 \cap W_2$.

   (b) Define the map $T : W_1 \times W_2 \to W_1 + W_2$ by $T(w_1, w_2) := w_1 + w_2$. Prove that T is a linear transformation.

   (c) Use the above to prove that $dim(W_1 + W_2) + dim(W_1 \cap W_2) = dim(W_1) + dim(W_2)$.

3. Let $\mathcal{P}_2[0, 2]$ represent the set of all polynomials with real coefficients and of degree less than or equal to 2, defined on $[0, 2]$. For $p := (p(t)) \in \mathcal{P}_2$ and $q := (q(t)) \in \mathcal{P}_2$, define

$$< p, q > := p(0)q(0) + p(1)q(1) + p(2)q(2).$$

   (a) Verify that $< p, q >$ is an inner product.

   (b) Let T represent the linear transformation that maps an element $p \in \mathcal{P}_2$ to the closest element of the span of the polynomials 1 and t in the sense of the norm associated with the inner product. Find the matrix A of T in the standard basis $\{1, t, t^2\}$ of $\mathcal{P}_2$.

   (c) Is A symmetric? Is T self-adjoint? Do these contradict each other?

   (d) Find the minimal polynomial of T.

4. Suppose that T is a linear map from a vector space V to $\mathbb{F}$ where $\mathbb{F}$ can be either $\mathbb{R}$ or $\mathbb{C}$. Prove that if a vector u in V is not in $null(T)$, then

$$V = null(T) \oplus \{\alpha u : \alpha \in \mathbb{F}\}$$

where $null(T)$ is the null space of T.

5. (a) Suppose $p, q \in \mathbb{R}$ with p and q positive and $1/p + 1/q = 1$. Show that for any matrix $A \in \mathbb{C}^{n\times n}$, we have $\|A\|_p = \|A^*\|_q$, where A^* is the conjugate transpose of A. Here $\|A\|_p$ denotes the matrix $p-$norm induced by the vector p-norm defined by $\|\mathbf{x}\|_p := (\sum_{i=1}^n |x_i|^p)^{1/p}$.

   (b) Prove that
$$\|A\|_2^2 \leq \|A\|_p \|A\|_q$$
   for any $A \in \mathbb{C}^{n\times n}$ and any positive p and $q \in \mathbb{R}$ with $1/p + 1/q = 1$.

   (c) Prove that for any $p \geq 1$ and any diagonal matrix $D \in \mathbb{C}^{n\times n}$, we have
$$\|D\|_p = \max\{|d_{ii}| : 1 \leq i \leq n\}\,.$$

   (d) Show that $\|A\|_2$ is the largest singular value of A.

6. (a) Let
$$x := \begin{bmatrix} 1 \\ 7 \\ 2 \\ 3 \\ -1 \end{bmatrix}, \qquad y := \begin{bmatrix} -4 \\ 4 \\ 4 \\ 0 \\ -4 \end{bmatrix}.$$
   Is there an orthogonal matrix Q so that $Qx = y$? If so, use EXACT arithmetic to find it. If not, explain why.

   (b) Let
$$A := \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}, \qquad b := \begin{bmatrix} 5 \\ -1 \\ 0 \end{bmatrix}.$$
   Compute a QR decomposition of A using Householder reflections and then solve the least square problem $\min_x \|b - Ax\|_2$ using the QR decomposition.

7. For the matrix $A := \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ -1/2 & \sqrt{3}/2 \\ \sqrt{3} & 1 \end{pmatrix}$, obtain the singular value decomposition of A (in the from $A = U\Sigma V^T$ where U and V are orthogonal and Σ is diagonal). Use this to find the Frobenius norm $\|A\|_F$ and the 2-norm $\|A\|_2$.

8. Suppose that $A \in \mathbb{C}^{n\times n}$ is normal, i.e., $AA^* = A^*A$. Show that if A is also upper triangular, it must be diagonal. Use this to show that A is normal if and only if it has n orthonormal eigenvectors.