University of Alabama System

Joint Ph.D. Program in Applied Mathematics

Joint Program Exam: Linear Algebra and Numerical Linear Algebra

May 2016

- This is a closed book exam. The duration of the exam is **three and an half hours**.
- You are required to do **7 out of the 8 problems** for full credit.
- Each problem is worth 10 points; multiple parts of a given problem have equal weights (unless otherwise specified).
- You must justify your solutions: cite theorems that you use, provide counter examples for disproving theorems, give explanations and show all the calculations for the numerical problems.
- Start each solution on a new page. Write the last four digits of your university **student ID number** and the problem number on every page (do not put your name). Write only on one side of the page.
- No calculators or other electronic devices are allowed.
- Please write legibly with a pen or a dark pencil.

1. Let $V = P_2(\mathbb{R})$ and T is defined by $T(ax^2 + bx + c) = cx^2 + bx + a$.

   (a) Determine if T is diagonalizable.

   (b) If T is diagonalizable, find a basis γ for V such that $[T]_\gamma$ is a diagonal matrix. What is $[T]_\gamma$?

2. Let $Q \in R^{n\times n}$ be an orthogonal matrix (i.e. $Q^TQ = QQ^T = I$).

   (a) For any $x \in R^n$, show that $\|Qx\|_2 = \|x\|_2$ and hence $\|Q\|_2 = 1$.

   (b) For any $A \in R^{n\times n}$, show that $\|QA\|_2 = \|A\|_2$.

   (c) For any $A \in R^{n\times n}$, define $B = Q^{-1}AQ$ (i.e. A and B are orthogonally similar), show that $\|B\|_2 = \|A\|_2$.

3. Let $A \in R^{n\times m}$, $n > m$ and $\text{rank}(A) = m$. The singular value decomposition (SVD) of A is $A = U\Sigma V^T$, where $U \in R^{n\times n}$ and $V \in R^{m\times m}$ are orthogonal, and $\Sigma \in R^{n\times m}$ has singular values $\sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_m > 0$.

   (a) Determine the SVD decompositions of the matrices

   $$(A^TA)^{-1}, \quad (A^TA)^{-1}A^T, \quad A(A^TA)^{-1}, \quad \text{and} \quad A(A^TA)^{-1}A^T$$

   in terms of the SVD of A. Please specify the dimensions and elements of the obtained Σ matrices.

   (b) Use the results of part (a) to determine the matrix 2−norms

   $$\|(A^TA)^{-1}\|_2, \quad \|(A^TA)^{-1}A^T\|_2, \quad \|A(A^TA)^{-1}\|_2, \quad \text{and} \quad \|A(A^TA)^{-1}A^T\|_2$$

   (c) For any matrix $A = (a_{ij})$, $A \in \mathbb{R}^{n\times m}$, define

   $$||A||_F = \left(\sum_{i=1}^{n}\sum_{j=1}^{m} a_{ij}^2\right)^{1/2}.$$

   This is the Frobenius matrix norm. Show that

   $$||A||_F = (\sigma_1^2 + \cdots + \sigma_m^2)^{1/2}$$

   where σ_i are the singular values of A.

4. Consider a least squares problem

   $$\begin{bmatrix} 2 & 3 \\ 2 & 4 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 7 \\ 3 \\ 1 \end{bmatrix},$$

   (a) Compute a QR decomposition of the matrix, with exact arithmetic, by using the Householder reflector method.

   (b) Compute the least squares solution based on the QR decomposition of part (a).

5. Given a matrix A of size $n \times n$, show that the following statements are equivalent:

   (a) A is normal ($A^*A = AA^*$);

(b) A is unitarily diagonalizable.

6. Assume that $A \in \mathbb{C}^{n\times n}$ is non-singular and has non-singular principal minors ($\det(A_k \neq 0)$) for all $k = 1, \ldots, n$. Then $A = LDM^*$ where the unique matrices L, M are unit lower triangular matrices and the unique matrix D is diagonal.

   (a) Assume A is Hermitian. Prove that

   $$A = LDL^*$$

   where L and D are a unique unit lower triangular matrix and a unique diagonal matrix, respectively.

   (b) Assume A is Hermitian. Prove that

   $$\text{A is positive definite} \quad \Longleftrightarrow \quad \det(A_k) > 0, \ \forall\ k = 1, \ldots, n.$$

   (c) Let A be Hermitian (symmetric) and positive definite. Then $A = GG^*$ where G is a unique lower triangular matrix with real positive diagonal entries. Given

   $$A = \begin{bmatrix} 1 & 3 & 0 & 0 \\ 3 & 13 & 8 & 0 \\ 0 & 8 & 25 & 15 \\ 0 & 0 & 15 & 41 \end{bmatrix}.$$

   Find its Cholesky decomposition: $A = GG^*$.

7. Given

   $$A = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}.$$

   (a) Use the Power Method to calculate the dominant eigenvalue and its corresponding eigenvector of A. Please do at least three iterations, carry out the calculation with three significant digits, and start with the vector $x_0 = [0\ 1]'$.

   (b) How would you revise the Power Method so that the algorithm could calculate the smallest eigenvalue and its corresponding eigenvector? Write down an algorithm.

   (c) Use part (b) to calculate the smallest eigenvalue and its corresponding eigenvector of A. Do two iterations with three significant digits starting with $x_0 = [1\ 0]'$.

8. Let $A \in \mathbb{R}^{n\times n}$ be symmetric and positive definite. Show that

   $$\det \begin{pmatrix} a_{11} & \ldots & a_{1n} & x_1 \\ & \ldots & & \vdots \\ a_{n1} & \ldots & a_{nn} & x_n \\ x_1 & \ldots & x_n & 0 \end{pmatrix} < 0$$

   for every nonzero vector $x = (x_1, \ldots, x_n)'$.