The University of Alabama System

Joint Ph.D Program in Applied Mathematics

Linear Algebra and Numerical Linear Algebra JP Exam

September 2015

Instructions:

- This is a closed book examination. Once the exam begins, you have three and one half hours to do your best. You are required to do **seven of the eight problems for full credit.**
- Each problem is worth 10 points; parts of problems have equal value unless otherwise specified.
- Justify your solutions: cite theorems that you use, provide counter examples for disproof, give explanations, and show calculations for numerical problems.
- Begin each solution on a new page and write the last four digits of your university **student ID number**, and problem number, on every page. Please write only on one side of each sheet of paper.
- The use of calculators or other electronic gadgets is not permitted during the exam.
- Write legibly using dark pencil or pen.

1. A square matrix N is called nilpotent if $N^m = 0$ for some positive integer m.

   (a) Is the sum of two nilpotent matrices nilpotent? If yes, prove it. If not, give a counter example.

   (b) Is the product of two nilpotent matrices nilpotent? If yes, prove it. If not, give a counterexample.

   (c) Prove that all eigenvalues of a nilpotent matrix are zero.

   (d) Prove that the only nilpotent matrix that is diagonalizable is the zero matrix.

2. Let $A \in \mathbb{R}^{m\times n}$ be a matrix.

   (a) Prove that if the matrix $A^T A$ is invertible, then the matrix
   $$I - A(A^T A)^{-1}A^T$$
   is symmetric positive semi-definite.

   (b) In addition, let $B \in \mathbb{R}^{m\times p}$. Prove that if $A^T A$ and $B^T B$ are invertible and if the ranges of A and B do not share a nontrivial subspace, then the matrix
   $$B^T(I - A(A^T A)^{-1}A^T)B$$
   is invertible.

3. Let $\lambda_1, \cdots, \lambda_n$ be eigenvalues of A, and A be diagonalizable such that $X^{-1}AX = D = diag(\lambda_1, \cdots, \lambda_n)$. Prove that if $A' = A + E$ and λ' is an eigenvalue of A', then
$$\min_{1\leq i\leq n} |\lambda' - \lambda_i| \leq \kappa_\infty(X)\|E\|_\infty.$$

4. (a) Let $A \in \mathbb{R}^{m\times n}$ and $b \in \mathbb{R}^m$. Suppose that $m > n$ and rank $(A) = n$. Show that the solution of
   $$\min_{x\in\mathbb{R}^n} \|Ax - b\|_2$$
   is equal to $\bar{x} = (A^T A)^{-1}A^T b$.

   (b) Suppose, on the other hand, that $m < n$ and rank $(A) = m$. Show that the solutions of the linear system $Ax = b$ form a translated $(n - m)$-dimensional subspace of $\mathbb{R}^n$. Give a formula for the minimum ℓ_2-norm solution of $Ax = b$.

5. (a) Use the Schur factorization to show that A is unitarily diagonalizable if and only if it is Hermitian.

   (b) Now suppose that $A \in \mathbb{C}^{m\times m}$ is Hermitian. Show that A is positive definite if and only if every eigenvalue of A is strictly positive.

6. (a) Establish that the matrix 2-norm $\|A\|_2$ is equal to the largest singular value of A.

   (b) Show that the matrix 2-norm and Frobenius norm are *unitarily invariant*:

$$\|A\|_2 = \|UAV\|_2 \qquad \|A\|_F = \|UAV\|_F$$

   for any $A \in \mathbb{C}^{m\times n}$ and unitary $U \in \mathbb{C}^{m\times m}$, $V \in \mathbb{C}^{n\times n}$.

7. Let V be an inner product space and $W \subset V$ a finite dimensional subspace with orthonormal basis $\{u_1, \ldots, u_n\}$. For every $x \in V$, define

$$P(x) = \sum_{i=1}^{n} < x, u_i > u_i \,.$$

   (a) Prove that $x - P(x) \in W^{\perp}$, hence P is the orthogonal projection onto W.

   (b) Prove that $\|x - P(x)\| \leq \|x - z\|$ for every $z \in W$, and that if $\|x - P(x)\| = \|x - z\|$ for some $z \in W$, then $z = P(x)$.

8. Let $A = \begin{bmatrix} 3 & -3 \\ 0 & 4 \\ 4 & -1 \end{bmatrix}$.

   (a) Find the QR factorization of A by Householder reflectors.

   (b) Use the results in (a) to find the least squares solution of $A\mathbf{x} = \mathbf{b}$, where $\mathbf{b} = [16 \;\; 11 \;\; 17]^T$.