UNIVERSITY OF ALABAMA SYSTEM

Joint Doctoral Program in Applied Mathematics

Joint Program Exam in Linear Algebra and Numerical Linear Algebra

May 2015

Exam Rules:

- This is a closed book examination. Once the exam begins, you have three and one half hours to do your best. **You are required to do seven of the eight problems for full credit.** If you answer all eight problems, your best seven will be graded.
- Each problem is worth 10 points; parts of problems have equal value unless otherwise specified.
- Justify your solutions: cite theorems that you use, provide counter examples for disproof, give explanations, and show calculations for numerical problems.
- Begin each solution on a new page and write the last four digits of your university student ID number, and problem number, on every page. Please write only on one side of each sheet of paper.
- If you are asked to prove a theorem, do not merely quote that theorem as your proof; instead, produce an independent proof.
- The use of calculators or other electronic gadgets is not permitted during the exam.
- Write legibly using dark pencil or pen.

1. (a) Let $A \in \mathbb{R}^{n\times n}$ be nonsingular, $b \in \mathbb{R}^n$ and $Ax = b$. Assume, all matrix norms below are induced matrix norms. Let $\hat{x}$ be a computed solution and let $r = b - A\hat{x}$, be the residual for $\hat{x}$. Show that

$$\frac{\|x-\hat{x}\|}{\|x\|} \leq \kappa(A)\frac{\|r\|}{\|b\|}, \tag{1}$$

where $\kappa(A)$ is the condition number of A.

   (b) Consider the problem

$$Ax = b, \quad A = \begin{bmatrix} 10^{-4} & 1 \\ 1 & 1 \end{bmatrix}, \quad b = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \tag{2}$$

in $p = 3$ decimal digits floating point arithmetic.

   i. Solve (2) using LU factorization without pivoting.
   ii. Solve (2) using LU factorization with partial pivoting (GEPP).
   iii. Relate your results to the estimate (1), use ∞−norm. On the basis of your computations, what is your conclusion regarding backward stability of the above two algorithms?

2. (a) Suppose that $A \in \mathbb{R}^{n\times n}$ is symmetric and nonsingular with *LU factorization* $A = LU$. Show that there exists a unique diagonal matrix $A \in \mathbb{R}^{n\times n}$ such that $A = LDL^T$.

   (b) Show that $A \in \mathbb{R}^{n\times n}$ is symmetric positive definite if and only if A has a *Cholesky factorization*: $A = BB^T$ where B is a lower triangular matrix with positive diagonal entries.

3. Consider the *linear least squares problem*

$$\min_x \|b - Ax\|_2, \quad A = \begin{bmatrix} 3 \\ 0 \\ 4 \end{bmatrix}, \; b = \begin{bmatrix} 10 \\ 5 \\ 5 \end{bmatrix}. \tag{3}$$

   (a) Compute the *QR factorization* of A using *Householder reflections.*
   (b) Solve the problem (3) using the *QR factorization method.*
   (c) Find the *pseudo-inverse* of A.

4. Suppose that $A \in \mathbb{C}^{n\times n}$ has a Schur factorization $A = QTQ^*$, where $Q \in \mathbb{C}^{n\times n}$ is unitary and $T \in \mathbb{C}^{n\times n}$ is upper triangular.

(a) Show that any Hermitian matrix A is unitarily diagonalizable with real eigenvalues.

(b) Show that if $A \in \mathbb{R}^{n \times n}$ is symmetric, then its largest eigenvalue λ_1 and smallest eigenvalue λ_n satisfy

$$\lambda_1 = \max_{x \neq 0,\, x \in \mathbb{R}^n} \frac{x^T A x}{x^T x}, \qquad \lambda_n = \min_{x \neq 0,\, x \in \mathbb{R}^n} \frac{x^T A x}{x^T x}.$$

5. (a) Suppose that $A \in \mathbb{R}^{n \times m}$ has *singular value decomposition* $A = U \Sigma V^*$. Derive an orthonormal basis for the column space, row space, and null space of A.

   (b) Use Part a. to establish the *Rank-nullity Theorem*: $\dim(\mathcal{R}(A)) + \dim(\mathcal{N}(A)) = m$.

6.

   (a) Assume $A \in \mathbb{R}^{n \times n}$ is diagonalizable and it has a dominant eigenvalue λ_1 with corresponding eigenvector x_1. Describe the *Power Method* for finding the dominant eigenpair (λ_1, x_1).

   (b) Apply the *Power Method* to

$$A = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix} \tag{4}$$

   to approximate the dominant eigenpair with the starting vector $x_0 = (0, 1)^T$. Do two iterations. Calculate $|\lambda_1 - \mu_2|$, where μ_k, $k = 1, 2, \ldots$ is an approximation to λ_1 calculated in the kth iteration of the *Power Method*.

   (c) For the matrix (4) prove convergence $\lim_{k \to \infty} \mu_k = \lambda_1$ of the *Power Method* and establish the asymptotic rate of convergence.

7. (a) Let $A = \begin{bmatrix} 2 & 0 & 0 \\ 3 & 2 & 0 \\ 0 & 0 & 4 \end{bmatrix}$. Determine a similarity transformation $A = PJP^{-1}$, where J is a Jordan form of A.

   (b) Let $L(x) = Ax$, where $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. Find a representation of L in the coordinate system E of eigenvectors of A. You must prove anything you want to use.

8. (a) Let V be a finite dimensional real inner product space, and let $T : V \to V$ be an *orthogonal operator* ($T^T T = T T^T = I$, and so T is an isometry) on V. Prove that if a subspace $W \subset V$ is invariant under T, i.e. $TW \subset W$, then so is its orthogonal complement $W^\perp$, i.e. $TW^\perp \subset W^\perp$. You must prove anything you want to use.

   (b) Let V be a finite dimensional real vector space, and let $T : V \to V$ be a *linear operator* on V. Prove that V has either one-dimensional or a two-dimensional invariant subspace $W \subset V$.