The University of Alabama System

Joint Ph.D Program in Applied Mathematics

Linear Algebra and Numerical Linear Algebra JP Exam

September 2014

Instructions:

- This is a closed book examination. Once the exam begins, you have three and one half hours to do your best. You are required to do **seven of the eight problems for full credit.**
- Each problem is worth 10 points; parts of problems have equal value unless otherwise specified.
- Justify your solutions: cite theorems that you use, provide counter examples for disproof, give explanations, and show calculations for numerical problems.
- Begin each solution on a new page and write the last four digits of your university **student ID number**, and problem number, on every page. Please write only on one side of each sheet of paper.
- The use of calculators or other electronic gadgets is not permitted during the exam.
- Write legibly using dark pencil or pen.

1. A real $n \times n$ matrix A is an *isometry* if it preserves length: $\|Ax\| = \|x\|$ for all vectors $x \in \mathbb{R}^n$. Show that the following are equivalent

   (a) A is an isometry

   (b) $< Ax, Ay >=< x, y >$ for all vectors x, y, so A preserves inner products

   (c) $A^{-1} = A^*$

   (d) The columns of A are unit vectors that are mutually orthogonal

2. Let $M_n(\mathbb{R})$ be the vector space of all $n \times n$ matrices with real entries, and $A \in M_n(\mathbb{R})$ be diagonalizable so that we have a nonsingular matrix $W \in M_n(\mathbb{C})$ and a diagonal matrix $\Lambda \in M_n(\mathbb{C})$, such that $A = W \Lambda W^{-1}$. Define

$$B = \begin{bmatrix} 0 & -A \\ 2A & 3A \end{bmatrix}$$

   Prove that B is diagonalizable and give the diagonalization of B (i.e. give the eigen-decomposition of B in terms of its eigenvalues and eigenvectors)
   (Hint: one can first consider the $n = 1$ case where $A = 1$)

3. Define $T \in \mathcal{L}(\mathbb{C}^n)$ by $T : (w_1, w_2, w_3, w_4)^t \mapsto (0, w_2 + w_4, w_3, w_4)^t$.

   (a) Determine the minimal polynomial of T

   (b) Determine the characteristic polynomial of T

   (c) Determine the Jordan form of T

4. Let V be a vector space, $T \in L(V, V)$ such that $T \circ T = T$. Prove that $\operatorname{Ker} T = \operatorname{Im}(I - T)$ and $V = \operatorname{Ker} T \oplus \operatorname{Im} T$.

5. Let x be a unit vector in $\mathbb{C}^n$. Define $H = I - 2xx^*$. Prove the following statements:

   (a) $Hx = -x$.

   (b) If y is orthogonal to x, then $Hy = y$.

   (c) The matrix H is Hermitian and unitary.

   (d) Explain why H can be interpreted as a reflection to the subspace

$$(\operatorname{span}\{x\})^{\perp} = \{y : x^* y = 0\}.$$

6. (a) Find the reduced QR factorization of $A = \begin{pmatrix} 1 & 3 \\ -2 & 4 \\ 2 & -5 \end{pmatrix}$.

   (b) use the result in part (a) to find

      i. the least squares solution of the system $A \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3 \\ -11 \\ -11 \end{pmatrix}$ and the corresponding residual vector; and

      ii. the orthogonal projector on the column space of A (without using A itself, but in terms only of the orthogonal factor of A).

7. For the matrix $A = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ -1/2 & \sqrt{3}/2 \\ \sqrt{3} & 1 \end{pmatrix}$, obtain the singular value decomposition of A (in the from $A = U\Sigma V^T$ where U and V are orthogonal and Σ is diagonal). Use this to find the Frobenius norm $\|A\|_F$ and the 2−norm $\|A\|_2$.

8. Suppose that A is a real, $n \times n$ symmetric matrix with $A^3 = A^2 + A - I$. Show that A is invertible and in fact A is its own inverse.