UNIVERSITY OF ALABAMA SYSTEM

Joint Doctoral Program in Applied Mathematics

Joint Program Exam in Linear Algebra and Numerical Linear Algebra

May 2014

Exam Rules:

- This is a closed book examination. Once the exam begins, you have three and one half hours to do your best. **You are required to do seven of the eight problems for full credit.**
- Each problem is worth 10 points; parts of problems have equal value unless otherwise specified.
- Justify your solutions: cite theorems that you use, provide counter examples for disproof, give explanations, and show calculations for numerical problems.
- Begin each solution on a new page and write the last four digits of your university student ID number, and problem number, on every page. Please write only on one side of each sheet of paper.
- If you are asked to prove a theorem, do not merely quote that theorem as your proof; instead, produce an independent proof.
- The use of calculators or other electronic gadgets is not permitted during the exam.
- Write legibly using dark pencil or pen

1. Let $A \in \mathbb{C}^{n\times n}$, and let

$$\det(\lambda I_n - A) = \lambda^n + a_{n-1}\lambda^{n-1} + \cdots + a_1\lambda + a_0$$

denote the characteristic polynomial of A (here I_n is the $n \times n$ identity matrix).

   (a) Prove that if A is invertible, then

$$A^{-1} = -\frac{1}{a_0}\left[A^{n-1} + a_{n-1}A^{n-2} + \cdots + a_2 A + a_1 I_n\right].$$

   (b) Use this formula to compute A^{-1} for

$$A = \begin{bmatrix} -2 & 1 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \end{bmatrix}$$

2. Find a reduced SVD for the matrix $A = \begin{bmatrix} 3 \\ 0 \\ 4 \end{bmatrix}$. Use it to solve the linear least squares (LS) problem

$$\min_x \|b - Ax\|_2, \qquad A = \begin{bmatrix} 3 \\ 0 \\ 4 \end{bmatrix}, \quad b = \begin{bmatrix} 10 \\ 5 \\ 5 \end{bmatrix}$$

3. Let $A \in \mathbb{C}^{n\times n}$ be a nonsingular upper Hessenberg matrix. Prove that if $A = LU$ is an LU decomposition of A, then $L = (l_{ij})$ has zeros below its first subdiagonal, i.e., $l_{ij} = 0$ for all $i \geq j + 2$.

4. Let $A \in \mathbb{C}^{n\times n}$ be a normal matrix, i.e., $AA^* = A^*A$. Use Schur decomposition to prove that there exists a unitary matrix Q and a diagonal matrix D such that $A = QDQ^*$.

5. Let $V \subset \mathbb{C}^n$ be a k-dimensional vector subspace, $k < n$. Let $\{q_1, \ldots, q_k\}$ be an orthonormal basis in V. Let $Q \in \mathbb{C}^{n\times k}$ denote the matrix whose columns are $q_1, \ldots, q_k$. Prove that

$$P = QQ^*$$

is the orthogonal projection of $\mathbb{C}^n$ onto V, i.e., the projection onto V along $V^\perp$.

6. (a) Let $A \in \mathbb{C}^{m\times n}$. Suppose $B = QA$, where $Q \in \mathbb{C}^{m\times m}$ is a unitary matrix. Show that A and B have the same singular values.

   (b) Let $B \in \mathbb{C}^{m\times n}$ be obtained by interchanging two rows of a matrix $A \in \mathbb{C}^{m\times n}$. Show that A and B have the same singular values.

   (c) Let $B \in \mathbb{C}^{m\times n}$ be obtained by interchanging two columns of a matrix $A \in \mathbb{C}^{m\times n}$. Show that A and B have the same singular values.

7. Suppose a matrix $A \in \mathbb{C}^{m\times n}$ with $m \geq n$ has the form $A = \begin{bmatrix} A_1 \\ A_2 \end{bmatrix}$, where A_2 is a nonsingular matrix of dimension $n \times n$ and A_1 is an arbitrary matrix of dimension $(m-n) \times n$. Let A^+ be the pseudoinverse of A defined as $A^+ = (A^*A)^{-1}A^*$. Prove that $\|A^+\|_2 \leq \|A_2^{-1}\|_2$.

8. Let $A \in \mathbb{C}^{m\times n}$ with $m \geq n$, and let $A = \hat{Q}\hat{R}$ be a reduced QR decomposition of A. Suppose $\hat{R}$ has exactly k nonzero diagonal entries ($k < n$). What does this imply about the rank of A? You should choose one of the following answers and prove that your answer is correct:

   (a) $\operatorname{rank} A = k$;

   (b) $\operatorname{rank} A \geq k$;

   (c) $\operatorname{rank} A \leq k$.

   (No need to show that the other two answers are wrong.)

Solutions:

1. (a) By the Cayley-Hamilton theorem,

$$A^n + a_{n-1}A^{n-1} + \cdots + a_2A^2 + a_1A + a_0I_n = 0_n$$

where 0_n denotes the zero $n \times n$ matrix. We also know that

$$a_0 = \det(-A) = (-1)^n \det A \neq 0,$$

because A is invertible. Now multiplying by A^{-1} and dividing by a_0 gives the desired result.

(b) We have

$$\det(\lambda I_3 - A) = \lambda^3 - \lambda^2 - 4\lambda + 4$$

therefore

$$\begin{aligned} A^{-1} &= -\frac{1}{4}\big(A^2 - A - 4I_3\big) \\ &= -\frac{1}{4}\left(\begin{bmatrix} 4 & -1 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 4 \end{bmatrix} - \begin{bmatrix} -2 & 1 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \end{bmatrix} - \begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix}\right) \\ &= -\frac{1}{4}\begin{bmatrix} 2 & -2 & -1 \\ 0 & -4 & 2 \\ 0 & 0 & -2 \end{bmatrix} = \begin{bmatrix} -0.5 & 0.5 & 0.25 \\ 0 & 1 & -0.5 \\ 0 & 0 & 0.5 \end{bmatrix}. \end{aligned}$$

Note: the students can use the Cayley-Hamilton theorem. They can also use the fact $a_0 \neq 0$ without proof.

2. We have

$$A^*A = \begin{bmatrix} 3 & 0 & 4 \end{bmatrix} \begin{bmatrix} 3 \\ 0 \\ 4 \end{bmatrix} = 9 + 0 + 16 = 25$$

therefore the only singular value is $\sigma = \sqrt{25} = 5$. The second unitary matrix V^* in the reduced SVD

$$A = \hat{U}\hat{D}V^*$$

is a 1×1 matrix, hence we can set it to $V^* = [1]$. The middle diagonal matrix is $\hat{D} = [5]$. Therefore the right matrix in the reduced SVD is

$$\hat{U} = AV\hat{D}^{-1} = \begin{bmatrix} 3 \\ 0 \\ 4 \end{bmatrix} \times \frac{1}{5} = \begin{bmatrix} 0.6 \\ 0 \\ 0.8 \end{bmatrix}$$

Now we solve the least squares problem by standard formula

$$x = V\hat{D}^{-1}\hat{U}^* b = \frac{1}{5}\begin{bmatrix} 0.6 & 0 & 0.8 \end{bmatrix} \begin{bmatrix} 10 \\ 5 \\ 5 \end{bmatrix} = 2.$$

Note: The students can use the SVD and the solution of the least square problem via normal equations without proof. They should prove the above standard formula, e.g., as follows:

$$x = (A^*A)^{-1}A^*b = (V\hat{D}\hat{U}^*\hat{U}\hat{D}V^*)^{-1}V\hat{D}\hat{U}^*b = V\hat{D}^{-1}\hat{U}^*b.$$

3. We have

$$H = LU$$

where H is the given upper Hessenberg matrix. Since H is nonsingular, so are both L and U. Thus

$$L = HU^{-1}$$

Here H is upper Hessenberg and U^{-1} is upper triangular. Therefore for $i \geq j+2$ we have (denoting by u'_{ij} the components of U^{-1})

$$l_{ij} = \sum_{k=1}^{n} h_{ik}u'_{kj} = 0$$

because $h_{ik} = 0$ for all $k \leq i-2$ and $u'_{kj} = 0$ for all $k \geq j+1$, in particular for all $k \geq i-1$.

4. A standard argument from the lecture.

5. For any vector $w \in \mathbb{C}^n$ we have $w = v + v^\perp$ where $v \in V$ and $v^\perp \in V^\perp$. Now

$$v = c_1 q_1 + \cdots + c_k q_k$$

where $c_i = \langle v, q_i \rangle$ for all $i = 1, \dots, k$. Since the rows of Q^* are vectors $q_1^*, \dots, q_k^*$, the product Q^*v is a vector with components

$$Q^*v = \begin{bmatrix} q_1^*v \\ \vdots \\ q_k^*v \end{bmatrix} = \begin{bmatrix} \langle v, q_1 \rangle \\ \vdots \\ \langle v, q_k \rangle \end{bmatrix} = \begin{bmatrix} c_1 \\ \vdots \\ c_k \end{bmatrix}$$

therefore

$$QQ^*v = \left[\begin{array}{c|c|c|c} q_1 & q_2 & \cdots & q_k \end{array}\right] \begin{bmatrix} c_1 \\ \vdots \\ c_k \end{bmatrix} = c_1 \begin{bmatrix} q_1 \end{bmatrix} + c_2 \begin{bmatrix} q_2 \end{bmatrix} + \cdots + c_k \begin{bmatrix} q_k \end{bmatrix} = v.$$

Similarly,

$$Q^*v^\perp = \begin{bmatrix} q_1^*v^\perp \\ \vdots \\ q_k^*v^\perp \end{bmatrix} = \begin{bmatrix} \langle v^\perp, q_1 \rangle \\ \vdots \\ \langle v^\perp, q_k \rangle \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix} = 0.$$

Putting it all together gives

$$QQ^*w = QQ^*(v + v^\perp) = v.$$

Note: It is enough for the students to show that $QQ^*v = v$ for every $v \in V$ and $QQ^*v^\perp = 0$ for every $v^\perp \in V^\perp$. This implies that QQ^* is the orthogonal projector onto V.

6. (a) If $A = UDV^*$ is an SVD for A, then

$$B = QA = QUDV^* = U_1DV^*$$

is an SVD for B, because $U_1 = QU$ is a unitary matrix.

(b) The operation of interchanging the rows i and j of a matrix A is equivalent to pre-multiplication of A by matrix

$$P = \begin{array}{c}
\begin{bmatrix}
1 & & & & & & & & & & \\
& \ddots & & & & & & & & & \\
& & 1 & & & & & & & & \\
& & & 0 & & & & 1 & & & \\
& & & & 1 & & & & & & \\
& & & & & \ddots & & & & & \\
& & & & & & 1 & & & & \\
& & & 1 & & & & 0 & & & \\
& & & & & & & & 1 & & \\
& & & & & & & & & \ddots & \\
& & & & & & & & & & 1
\end{bmatrix}
\begin{array}{l}
\\ \\ \\ \leftarrow i \\ \\ \\ \leftarrow j \\ \\ \\ \\ \\
\end{array}
\\
\begin{array}{ccccccccccc}
& & & \uparrow & & & & \uparrow & & & \\
& & & i & & & & j & & &
\end{array}
\end{array}$$

That is, $B = PA$. Since P is a unitary matrix, by part (a) we conclude that B and A have the same singular values.

(c) Just like in part (a), we note that if $B = AQ$ with a unitary matrix Q, then A and B have the same singular values. Then note that the operation of interchanging the columns i and j of a matrix A is equivalent to post-multiplication of A by the above matrix P, i.e., $B = AP$. Since P is unitary, A and B have the same singular values.

Note: The students can use the SVD without proof.

7. For any $x \in \mathbb{C}^m$, denote $A^+x = a$. Then $A^*x = A^*Aa$. From this, we get $a^*A^*x = a^*A^*Aa$. By the Cauchy-Schwartz inequality we get

$$\|Aa\|_2^2 = a^*A^*Aa = (Aa)^*x = \langle x, Aa\rangle \le \|Aa\|_2\|x\|_2$$

and hence $\|Aa\|_2 \le \|x\|_2$. Note that

$$Aa = \begin{bmatrix} A_1a \\ A_2a \end{bmatrix}.$$

Thus we have $\|A_2a\|_2 \le \|Aa\|_2 \le \|x\|_2$. Therefore $\|A_2a\|_2 \le \|x\|_2$. Note also that

$$\|a\|_2 = \|A_2^{-1}A_2a\|_2 \le \|A_2^{-1}\|_2\|A_2a\|_2.$$

Now $\|a\|_2 \leq \|A_2^{-1}\|_2 \|x\|_2$. Therefore, for any $x \in \mathbb{C}^m$ we have $\|A^+ x\|_2 \leq \|A_2^{-1}\|_2 \|x\|_2$, which implies $\|A^+\|_2 \leq \|A_2^{-1}\|_2$.

8. The correct answer is (b), i.e.,

$$\operatorname{rank} A \geq k.$$

Indeed, let $A = QR$ be the corresponding full QR decomposition obtained by adding $m-n$ columns to $\hat{Q}$ and $m-n$ rows of zeros to $\hat{R}$. Since Q is a unitary matrix, it is invertible, hence multiplication by Q does not change the rank of a matrix. Thus we have

$$\operatorname{rank} A = \operatorname{rank} R.$$

The rank of R is the number of linearly independent columns of R. If we examine the columns of R from left to right, every non-zero diagonal entry of R provides one more column linearly independent from the previous ones. Thus their total number is *at least* k.