University of Alabama System

Joint Ph.D Program in Applied Mathematics

Joint Program Exam: Linear Algebra and Numerical Linear Algebra

September 2012

Instructions:

- This is a closed book examination. Once the exam begins, you have three and one half hours to do your best. You are required to do **seven of the eight problems for full credit.**
- Each problem is worth 10 points; parts of problems have equal value unless otherwise specified.
- Justify your solutions: cite theorems that you use, provide counter examples for disproof, give explanations, and show calculations for numerical problems.
- Begin each solution on a new page and write the last four digits of your university **student ID number**, and problem number, on every page. Please write only on one side of each sheet of paper.
- The use of calculators or other electronic gadgets is not permitted during the exam.
- Write legibly using dark pencil or pen.

1. Let

$$A = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix},$$

where A is to be considered as a matrix over $\mathbb{C}$.

   (a) Determine the minimal and characteristic polynomials of A and the Jordan form of A.

   (b) Determine all generalized eigenvectors of A and a basis $\mathcal{B}$ of $\mathbb{C}^4$ with respect to which the operator $T_A : x \to Ax$ has Jordan form.

2. Let $\mathcal{P}_n$ be the vector space of all polynomials of degree at most n over $\mathbb{R}$. Define $T : \mathcal{P}_n \to \mathcal{P}_n$ by $T(p(x)) = xp'(x) - p(x)$.

   (a) (3 pts) Show that T is a linear transformation on $\mathcal{P}_n$.

   (b) (7 pts) Find the Null(T) and Range(T).

3. (a) Let $A \in \mathbb{R}^{n\times n}$ be a symmetric $n \times n$ matrix. Prove that A is positive definite, i.e., $x^T Ax > 0$ for all $x \in \mathbb{R}^n \setminus \{0\}$, if and only if all the eigenvalues of A are positive.

   (b)

$$A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & -1 \\ 0 & -1 & 3 \end{bmatrix}.$$

   Put $V = \mathbb{R}^3$. Define the map $* : V \times V \to \mathbb{R}$ by $u * v = u^T Av$ for all $u, v \in V$. Prove that $*$ is an inner product on V.

   (c) Use the inner product from above and the Gram-Schmidt orthogonalization process to find an orthonormal basis for V.

4. Let $A \in \mathbb{R}^{n\times n}$ be a symmetric positive definite matrix. Show that its 2-norm condition number can be computed as follows:

$$\text{cond}_2(A) = \frac{\max_{\|x\|_2=1} \langle Ax, x\rangle}{\min_{\|x\|_2=1} \langle Ax, x\rangle}$$

5. Let $U \in \mathbb{R}^{n\times n}$ be an upper triangular matrix with non-zero diagonal entries $u_{ii} \neq 0$. Show that its 2-norm condition number satisfies

$$\text{cond}_2(U) \geq \frac{\max_{1\leq i\leq n} |u_{ii}|}{\min_{1\leq i\leq n} |u_{ii}|}$$

6. Let $A, B \in \mathbb{C}^{n\times n}$ be two Hermitian matrices, and assume that B is positive definite. A number $\lambda \in \mathbb{C}$ is called a generalized eigenvalue for the pair (A, B) if $Ax = \lambda Bx$ for some non-zero vector $x \in \mathbb{C}^n$, in which case x is called generalized eigenvector. Prove that all the generalized eigenvalues are real and there exists a basis of generalized eigenvectors.

7. The matrix $A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ represents clockwise rotation by 90 degrees. Show that A can be factored into the product of two Householder reflectors, i.e. $A = Q_1 Q_2$, where $Q_i = I - 2v_i v_i^*$ and $||v_i||_2 = 1$ for $i = 1, 2$ by determining the vectors v_1, v_2.

8. Let $A \in \mathbb{C}^{n\times n}$. The power method to find the dominant eigenvalue and the corresponding eigenvector of the matrix A is as follows:
For $k = 1, 2, \ldots$, do

   set $w^{(k)} = Av^{(k-1)}$

   find the smallest integer p with $1 \leq p \leq n$ and $\|w^{(k)}\|_\infty = |w_p^{(k)}|$

   set $\mu_k = w_p^{(k)}$

   set $v^{(k)} = w^{(k)}/\mu_k$.

   Assume the following three conditions: (i) A has n linearly independent eigenvectors, x_k, $1 \leq k \leq n$, (ii) The eigenvalues λ_k satisfy

$$|\lambda_1| > |\lambda_2| \geq \cdots \geq |\lambda_n|$$

   and (iii) The vector $v^{(0)} \in \mathbb{C}^n$ is such that $v^{(0)} = \sum_{k=1}^n \xi_k x_k$ and $\xi_1 \neq 0$.
   Prove that

$$\lim_{k\to\infty} \mu_k = \lambda_1 , \quad |\lambda_1 - \mu_k| = O\left(\left|\frac{\lambda_2}{\lambda_1}\right|^k\right) .$$