UNIVERSITY OF ALABAMA SYSTEM

Joint Doctoral Program in Applied Mathematics

Joint Program Exam in Linear Algebra and Numerical Linear Algebra

TIME: THREE AND A HALF HOURS

September 2010

Instructions: Do 7 out of 8 problems. Include all work. Write your student ID number and problem number on every page.

1. Let $A \in \mathbb{R}^{n\times n}$ be a nonsingular matrix. Show that

$$\min\left\{\frac{\|\delta A\|_2}{\|A\|_2} \mid A+\delta A \text{ is singular}\right\} = \frac{1}{\kappa_2(A)}.$$

That is the relative distance to the nearest singular matrix is $1/\kappa_2(A)$. Here $\kappa_2(A)$ is the 2-norm condition number of a matrix A defined to be $\kappa_2(A) = \|A\|_2\|A^{-1}\|_2$.

2. Let $A \in \mathbb{R}^{n\times n}$ be of full rank, and let X be a matrix that diagonalizes A, i.e. $X^{-1}AX = D$, where D is a diagonal matrix with eigenvalues λ_i, $i = 1, \ldots, n$ of A on the diagonal. If $A' = A + E$, and λ' is an eigenvalue of A', prove that

$$\min_{1\le i\le n} |\lambda' - \lambda_i| \le \kappa_2(X)\|E\|_2.$$

3. (a) Let $A \in \mathbb{R}^{n\times n}$ be an orthogonal matrix. Prove that there exist $0 \le k \le n/2$, $\theta_i \in [0, 2\pi)$ for $i = 1, \ldots, k$ and an orthogonal matrix Q such that

$$Q^{-1}AQ = \begin{bmatrix} \begin{pmatrix} \cos\theta_1 & -\sin\theta_1 \\ \sin\theta_1 & \cos\theta_1 \end{pmatrix} & & & & & 0 \\ & \ddots & & & & \\ & & \begin{pmatrix} \cos\theta_k & -\sin\theta_k \\ \sin\theta_k & \cos\theta_k \end{pmatrix} & & & \\ & & & \pm 1 & & \\ & & & & \ddots & \\ 0 & & & & & \pm 1 \end{bmatrix}.$$

(b) For $n = 2$, interpret (a) geometrically.

4. Show that the LU decomposition without pivoting fails for the matrix

$$A = \begin{bmatrix} 2 & 4 & -2 & -2 \\ 1 & 2 & 4 & -3 \\ -3 & -3 & 8 & -2 \\ -1 & 1 & 6 & -3 \end{bmatrix}$$

Find the LU factorization of A by using partial pivoting.

5. Let $\delta = 10^{-6}$ and consider the overdetermined system $Ax = b$ given as

$$\begin{bmatrix} 1 & -1 \\ 0 & \delta \\ 0 & 0 \end{bmatrix} x = \begin{bmatrix} 0 \\ \delta \\ 1 \end{bmatrix}$$

(a) What is the least squares solution (by hand) to this overdetermined problem using the normal equations?

(b) If you compute the least squares solution to $Ax = b$ using the normal equations on a computer with $\epsilon_{\text{machine}} = 10^{-10}$, what result would you expect?

(c) The ∞-norm condition number of a matrix A is defined to be $\kappa_\infty(A) = \|A\|_\infty \|A^{-1}\|_\infty$. Compute the ∞-norm condition number of the coefficient matrix in the normal equations. Thus comment on the stability of using the normal equations for solving this least squares problem.

6. Let $A = \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}$ with a real $\lambda \in \mathbb{R}$. For which values of λ there is a 2×2 real matrix $B \in \mathbb{R}^{2\times 2}$ such that $A = B^2$? If there is such a matrix B, then how many distinct matrices B satisfy $A = B^2$?

7. Let A be a 6×6 complex matrix with characteristic polynomial $C_A(x) = (x^2+4)^3$. Assume that the dimensions of its eigenspaces are $\dim E_{2i} = 2$ and $\dim E_{-2i} = 1$. Find the minimal polynomial of A.

8. Apply the QR algorithm (without shift) to the matrix

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

Does it converge and produce the eigenvalues of A? If not, why? Apply the QR algorithm with the Rayleigh quotient shift. Does it help the convergence? Why?