University of Alabama System

Joint Ph.D Program in Applied Mathematics

Joint Program Exam: Linear Algebra and Numerical Linear Algebra

May 2010

Exam Rules:

- This is a closed book examination. Once the exam begins, you have three and one half hours to do your best. You are required to do seven of the eight problems for full credit.
- Each problem is worth 10 points; parts of problems have equal value unless otherwise specified.
- Justify your solutions: cite theorems that you use, provide counter examples for disproof, give explanations, and show calculations for numerical problems.
- Begin each solution on a new page and write your student ID number, and problem number, on every page. Please write only on one side of each sheet of paper.
- Write legibly using dark pencil or pen.

1. If u and v are m-vectors, the matrix $A = I + uv^*$ (where v^* represents the conjugate transpose of the vector v) is known as *rank-one perturbation of the identity.*

   (a) Show that if A is nonsingular, then its inverse has the form $A^{-1} = I + \alpha uv^*$ for some scalar α, and give an expression of α.

   (b) For what u and v is A singular? If it is singular, find a basis for the null space of A.

2. Suppose that $n = \dim(V) < \infty$ and $S, T \in \mathcal{L}(V)$, where $\mathcal{L}(V)$ denotes the space of all linear transformations from V to itself.

   (a) Prove that ST and TS have the same eigenvalues.

   (b) Suppose that T has n distinct eigenvalues and that S has the same eigenvectors as T (though not necessarily with the same eigenvalues). Prove that $ST = TS$.

3. Let $A = \begin{bmatrix} 3 & -3 \\ 0 & 4 \\ 4 & -1 \end{bmatrix}$.

   (a) Find the QR factorization of A by Householder reflectors.

   (b) Use the results in (a) to find the least squares solution of $A\mathbf{x} = \mathbf{b}$, where $\mathbf{b} = [16 \;\; 11 \;\; 17]^T$.

4. Consider the matrix

$$A = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}.$$

   Let $b \in \mathbb{R}^2$ be a unit vector and let $\delta b \in \mathbb{R}^2$ be a small perturbation vector, with $\|\delta b\| = \epsilon > 0$. Let x solve $Ax = b$ and let δx be the error associated with the perturbation δb, which satisfies $A(x + \delta x) = b + \delta b$, where $\|\cdot\|$ denotes the $2-$ norm.

   (a) Calculate the $2-$ norm condition number of A.

   (b) Determine the least upper bound on the relative error $\frac{\|\delta x\|}{\|x\|}$.

   (c) Find b and δb for which this upper bound is achieved (subject to $\|b\| = 1$ and $\|\delta b\| = \epsilon$).

5. Let A be a complex $n \times n$ Hermitian matrix, $\lambda \in \mathbb{R}$, and $\epsilon > 0$. Suppose there exists $v \in \mathbb{C}^n$ such that $\|v\| = 1$ and

$$\|Av - \lambda v\| < \epsilon,$$

where $\|\cdot\|$ denotes the 2$-$ norm. Prove that A has an eigenvalue λ' such that $|\lambda - \lambda'| < \epsilon$.

6. Let A be a real, symmetric $n \times n$ matrix. Let $f(x) = (x-2)^2(x+5)^3$ and suppose that $f(A) = 0$ and the trace of A is 0.

   (a) Show that $n = 7k$ for some $k \geq 1$, i.e., n is a multiple of 7.

   (b) Determine the minimal polynomial of A.

   (c) Determine the characteristic polynomial of A in terms of k.

   (d) Determine the trace of A^2 in terms of k.

7. Let V and W be finite dimensional Hermitian vector spaces with inner products $<\cdot,\cdot>_V$ and $<\cdot,\cdot>_W$, respectively. Let $T : V \to W$ be a linear transformation with adjoint T^*. Let B_0 be a basis for the null space (kernel) of T, and B_I be a basis for the image (range) of T^*. Prove that $B_0 \cup B_I$ is a basis for V.

8. Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ -1 & 3 & -2 \\ 4 & 5 & 6 \end{bmatrix}$ and let $A = U\Sigma V^T$ be the singular value decomposition of A, where the matrices U, Σ and V are defined as follows:

$$U = \begin{bmatrix} 0.3374 & -0.0285 & -0.7744 & -0.5345 \\ 0.4923 & -0.0071 & -0.3387 & 0.8018 \\ -0.0155 & 0.9989 & -0.0346 & 0.0000 \\ 0.8022 & 0.0357 & 0.5327 & -0.2673 \end{bmatrix}$$

$$\Sigma = \begin{bmatrix} 10.9234 & 0 & 0 \\ 0 & 3.7417 & 0 \\ 0 & 0 & 0.8239 \\ 0 & 0 & 0 \end{bmatrix}$$

$$V = \begin{bmatrix} 0.4162 & -0.2403 & 0.8769 \\ 0.5599 & 0.8276 & -0.0390 \\ 0.7164 & -0.5073 & -0.4790 \end{bmatrix}.$$

   (a) Is the matrix A^TA nonsingular? Explain why. If so, what is the inverse?

   (b) Exploit the SVD to find a basis for the $N(A^T)$ (i.e., the kernel of A^T) and the condition number $\kappa_2(A^TA)$ of A^TA in the 2-norm.