UNIVERSITY OF ALABAMA SYSTEM
Joint Doctoral Program in Applied Mathematics
Joint Program Exam: Linear Algebra and Numerical Linear Algebra

TIME: THREE AND ONE HALF HOURS

September, 2009

Instructions: Do 7 of the 8 problems for full credit. Include all work. Write your student ID number, and problem number on every page.

1. Let $\beta = (u_1, u_2, ..., u_n)$, $n \geq 2$, be a basis for a vector space V over the complex numbers, and define another vector $u_{n+1} = u_1 + u_2 + ... + u_n$. Prove that for each vector x in V there exist unique scalars $c_1, c_2, ..., c_{n+1}$ such that $c_1 + c_2 + ... + c_{n+1} = 1$ and $x = c_1 u_1 + c_2 u_2 + ... + c_{n+1} u_{n+1}$.

2. Let V be a vector space, and T be a linear operator on V such that $T \circ T = T$. Prove that $\operatorname{Ker} T = \operatorname{Im}(I - T)$ and $V = \operatorname{Ker} T \oplus \operatorname{Im} T$.

3. Let V be an n-dimensional vector space over the real numbers, and let T be a linear operator on V with n distinct eigenvalues.

   (a) Prove: If X is a linear operator on V such that $TX = XT$ then X is diagonalizable.

   (b) Let T be invertible and let Y be a linear operator on V such that $YT = T^{-1}Y$. Prove that the operator Y^2 is diagonalizable, and give an example to show that it is possible that Y is not diagonalizable in V.

4. Let
$$A = \begin{pmatrix} 1 & 0 & a & b \\ 0 & 1 & 0 & 0 \\ 0 & c & 3 & -2 \\ 0 & d & 2 & -1 \end{pmatrix}.$$

   (a) Determine conditions on a, b, c, and d so that there is only one Jordan block for each eigenvalue of A in the Jordan canonical form of A.

   (b) Suppose now $a = c = d = 2$ and $b = -2$. Find the Jordan canonical form of A.

5. Let $A = \begin{pmatrix} 1 & 1 \\ 1 & -2 \\ 1 & 3 \\ 1 & 0 \end{pmatrix}$.

   (a) Find a reduced QR factorization of A by the Gram-Schmidt process.

   (b) Use the QR factorization from (a) to find the least squares fit by a linear function for $(1, -2)$, $(-2, 0)$, $(3, 2)$ and $(0, 3)$.

6. Suppose that $A = (a_{ij}) \in \mathbb{C}^{n \times n}$ is normal, i.e., $AA^* = A^*A$. Show that if A is also upper triangular, it must be diagonal. Use this to show that A is normal if and only if it has n orthonormal eigenvectors.

   Hint: You may use the Schur decomposition.

7. Suppose that $A \in \mathbb{R}^{m \times n}$. Let $A = U\Sigma V^T$ be the SVD of A, $U \in \mathbb{R}^{m \times m}$, $V \in \mathbb{R}^{n \times n}$, $\Sigma = \operatorname{diag}(\sigma_1, ..., \sigma_p) \in \mathbb{R}^{m \times n}$, and $p = \min\{m, n\}$.

(a) Determine $\|A\|_2$ using the SVD of A.

(b) Determine an eigendecomposition of $A^T A$ in terms of the SVD of A.

(c) Determine an eigendecomposition of AA^T in terms of the SVD of A.

(d) Let $A = \begin{bmatrix} 3 \\ 0 \\ 4 \end{bmatrix}$

  (i) Find the singular values and singular vectors of A. (Hint: use (b) and (c)).

  (ii) Express A as the SVD form $A = U\Sigma V^T$.

8. Let $A \in \mathbb{C}^{n\times n}$ be nonsingular. Let $A = Q_1 R_1$ be a QR decomposition of A, and for $k \geq 1$ define inductively $AQ_k = Q_{k+1}R_{k+1}$, a QR decomposition of AQ_k.

   (a) Prove that there exists an upper triangular matrix U_k such that $Q_k = A^k U_k$ and a lower triangular matrix L_k such that $Q_k = (A^*)^{-k} L_k$.

   (b) Suppose $\lim_{k\to\infty} R_k = R_\infty$ and $\lim_{k\to\infty} Q_k = Q_\infty$ exist. Determine the eigenvalues of A in terms of R_∞.