UNIVERSITY OF ALABAMA SYSTEM
Joint Doctoral Program in Applied Mathematics
Joint Program Exam: Linear Algebra and Numerical Linear Algebra

TIME: THREE AND ONE HALF HOURS

May 2009

Instructions: Do 7 of the 8 problems for full credit. Include all work. Write your student ID number, and problem number, on every page.

1. Let V be a finite dimensional vector space and $T : V \to V$ a nonzero linear operator. Show that if $Ker(T) = Im(T)$, the dim V is an even integer and the minimal polynomial of T is $m(x) = x^2$.

2. (a) Let A be an $n \times n$ matrix such that the sum of its components in every row is equal to r. Show that r is an eigenvalue of A.

   (b) Let A be an $n \times n$ matrix of the form $A = aI + bJ$, where $a, b \in \mathbb{R}$, and I is the identity matrix, and J is the 'all one' matrix (all entries are one). Find all eigenvalues, eigenspaces and the characteristic polynomial of A.

3. Let V be a vector space, and $T : V \mapsto V$ be a linear operator on V such that $T \circ T = T$. Prove that $\operatorname{Ker} T = \operatorname{Im}(I - T)$ and $V = \operatorname{Ker} T \oplus \operatorname{Im} T$, where I is the identity and $\oplus$ means direct sum.

4. Let

$$A = \begin{bmatrix} 2 & 3 & 3 & 5 \\ 3 & 2 & 2 & 3 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

   (a) Find the eigenvalues λ and the associated eigenspace E_λ.

   (b) Determine if A is diagonalizable. If so, give matrices P, B such that $P^{-1}AP = B$ and B is diagonal. If not, explain carefully *why* A is not diagonalizable.

5. Let V be an inner product space and $W \subset V$ a finite dimensional subspace with ONB $\{u_1, \ldots, u_n\}$. For every $x \in V$ define $P(x) = \sum_{i=1}^{n} \langle x, u_i \rangle u_i$.

   (a) Prove that $x - P(x) \in W^\perp$.

   (b) Prove that P is the orthogonal projection on W.

   (c) Prove that $\|x - P(x)\| \leq \|x - z\|$ for every $z \in W$, and that if $\|x - P(x)\| = \|x - z\|$ for some $z \in W$, then $z = P(x)$.

6. Find the reduced QR factorization of

$$A = \begin{bmatrix} 1 & 3 \\ -2 & 4 \\ 2 & -5 \end{bmatrix}.$$

   Use the result to find

   (a) the least squares solution of the system $A\mathbf{x} = \mathbf{b}$ where $\mathbf{b} = [3, -11, -11]^T$ and the corresponding residual vector

   (b) the pseudo-inverse of A;

(c) the orthogonal projector on the column space of A (without using A itself, but in terms only of the orthogonal factor of A).

7. Let $A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 2 & 1 \end{bmatrix}$. Suppose the power method is applied with starting vector $\mathbf{x}_0 = [1, 1, -1]^T/\sqrt{3}$

(a) Determine whether or not the iteration will converge to an eigenpair of A, and if so, which one. Assume exact arithmetic.

(b) Repeat (a), except we now use inverse iteration using the same starting vector $\mathbf{x}_0$ and the Rayleigh quotient of $\mathbf{x}_0$ as approximation for the eigenvalue.

(c) Now answer both (a) and (b) again, except this time use standard fixed precision floating point arithmetic, i.e., computer arithmetic, and comment on the results.

8. Let $A \in \mathbb{C}^{n \times n}$ and x a unit eigenvector of A corresponding to eigenvalue λ. Let y be another unit vector and $\sigma =< Ay, y >$.

(a) Show that

$$|\lambda - \sigma| \leq 2\|A\|_2 \|y - x\|_2 \,.$$

(b) If A is Hermitian, show that there is a constant $C = C(A)$ such that

$$|\lambda - \sigma| \leq C(A) \|y - x\|_2^2 \,.$$