UNIVERSITY OF ALABAMA SYSTEM
Joint Doctoral Program in Applied Mathematics
Joint Program Exam: Linear Algebra and Numerical Linear Algebra

TIME: THREE AND ONE HALF HOURS

September, 2008

Instructions: Do 7 of the 8 problems for full credit. Include all work. Write your student ID number on every page of your exam.

1. Let V and W be vector spaces over a field F, and let $S : V \to W$ and $T : V \to W$ be linear transformations.

   (a) Prove that range$(S + T)$ is a subspace of range(S) +range(T).

   (b) Prove that rank$(S + T) \leq$ rank(S) +rank(T) when either V or W is finite-dimensional.

   (c) Use (b) to prove that rank$(A+B) \leq$ rank(A) +rank(B) when when A and B are $m \times n$ matrices over F.

2. Let $A \in \mathbb{C}^{2\times 2}$. Prove that $\lim_{n\to\infty} \|A^n\|_2 = 0$ if and only if $\rho(A) < 1$, where $\rho(A) = \max\{|\lambda_i|: \ \lambda_i$ is an eigenvalue of $A\}$ is the spectral radius of the matrix A.

3. Let $A \in \mathbb{R}^{n\times n}$ be nonsingular. Prove that if for any norm $\frac{\|\delta A\|}{\|A\|} < \frac{1}{\kappa(A)}$, then $A+\delta A$ is nonsingular. Here δA is a perturbation of A, and $\kappa(A)$ is the condition number of A.

4. (a) Show that a hermitian matrix $A \in \mathbb{C}^{n\times n}$ satisfies $x^*Ay = \overline{y^*Ax}$ for all $x, y \in \mathbb{C}^n$, and use the result to prove that x^*Ax and all eigenvalues of A are real. Here x^* stands for the conjugate transpose of x.

   (b) Show first that the eigenvalues of a unitary matrix are complex numbers with absolute value 1, then use this result and that of (a) to prove that a unitary, hermitian and positive definite square matrix is the identity matrix.

5. (a) Let $x, y \in \mathbb{R}^n$ such that $x \neq y$ but $\|x\|_2 = \|y\|_2$, show that there exists a reflector Q of the form $Q = I - 2uu^T$, where I is the $n \times n$ identity matrix, $u \in \mathbb{R}^n$, and $\|u\|_2 = 1$ such that $Qx = y$.

   (b) Let $A = \begin{bmatrix} 3 \\ 0 \\ 4 \end{bmatrix}$, $b = \begin{bmatrix} 10 \\ 5 \\ 5 \end{bmatrix}$, compute a reduced QR decomposition of A using Householder reflections and then solve the least square problem $\min_x \|b-Ax\|_2$ and calculate error $\|b - Ax\|_2$.

6. Let $A = I - \frac{1}{n}\mathbf{1}\mathbf{1}^T$, where I is the $n \times n$ identity matrix, and $\mathbf{1}$ is an n-vector, all of whose entries are equal to 1. Prove that the singular values of A are $1, 1, \ldots, 1, 0$.

7. Let $A = QTQ^*$ be a Schur decomposition of the matrix

$$A = \begin{bmatrix} 0 & -3 \\ 3 & 0 \end{bmatrix}.$$

   Find such a matrix T.

8. Apply the QR algorithm (without shift) to the matrix

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

Does it converge and produce the eigenvalues of A? If not, why? Apply the QR algorithm with the Rayleigh quotient shift. Does it help the convergence? Why?