UNIVERSITY OF ALABAMA SYSTEM

Joint Doctoral Program in Applied Mathematics

Joint Program Exam in Linear Algebra and Numerical Linear Algebra

TIME: THREE AND A HALF HOURS

May 2008

Instructions: Do 7 out of 8 problems. Include all work. Write your student ID number and problem number on every page.

1. Find all possible Jordan canonical forms for the following:
(a) A linear operator T with characteristic polynomial $\Delta(x) = (x-2)^4(x-3)^2$ and minimal polynomial $m(x) = (x-2)^2(x-3)^2$.
(b) A linear operator T with characteristic polynomial $\Delta(x) = (x-4)^5$ and such that $\dim \operatorname{Ker}(T-4I) = 3$.
Provide a complete list of all matrices satisfying the above requirements, up to the order of Jordan blocks. Explain your answers.

2. A *projector* is a square matrix P that satisfies $P^2 = P$. A projector P is an *orthogonal projector* if its kernel, $\operatorname{Ker} P$, is orthogonal to its range, $\operatorname{Range} P$. Let $P \in \mathbb{C}^{m\times m}$ be a nonzero projector. Show that $\|P\|_2 \geq 1$, with equality if and only if P is an orthogonal projector.

3. Let $A \in \mathbb{C}^{m\times m}$ be a Hermitian matrix. Prove that $r(x) = \frac{x^*Ax}{x^*x}$ is a real number for every $0 \neq x \in \mathbb{C}^m$. Prove that the range of the function $r(x)$ is the closed interval $[\lambda_{\min}, \lambda_{\max}]$, where $\lambda_{\min}$ and $\lambda_{\max}$ denote the minimum and maximum eigenvalues of A.

4. (a) Compute the condition numbers κ_1 and κ_∞ for the matrix

$$A = \begin{pmatrix} 5 & 1.001 \\ 10 & 2 \end{pmatrix}.$$

(b) Prove that $\kappa_1(A) = \kappa_\infty(A)$ for every nonsingular 2×2 matrix A.
(c) Prove that if $\kappa(A) = \|A\| \cdot \|A^{-1}\|$ is defined by any matrix norm (induced by a vector norm), then $\kappa(AB) \leq \kappa(A)\kappa(B)$ for any $n\times n$ invertible matrices.

5. Let $A \in \mathbb{R}^{n\times n}$ be an invertible matrix, $b \in \mathbb{R}^n$, and x denote the unique exact solution of the linear system $Ax = b$. Let also

$$(A + \delta A)\, y = b + \delta b.$$

Assume that

$$\frac{\|\delta A\|}{\|A\|} < \varepsilon, \qquad \frac{\|\delta b\|}{\|b\|} < \varepsilon,$$

and $\varepsilon\kappa(A) = r < 1$. Show that $A + \delta A$ is invertible and

$$\frac{\|y\|}{\|x\|} \leq \frac{1+r}{1-r}.$$

6. Suppose A is a 202×202 real matrix with $\|A\|_2 = 100$ and $\|A\|_F = 101$, where

$$\|A\|_F = \left(\sum_{i=1}^{202} \sum_{j=1}^{202} a_{ij}^2 \right)^{1/2}$$

denotes the Frobenius norm of A. Find the smallest possible value for the 2-norm condition number $\kappa_2(A)$.

7. Consider the linear least squares problem

$$\min_x \|Ax - b\|_2^2 \qquad \text{with} \quad A = \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix} \quad \text{and} \quad b = \begin{pmatrix} 5 \\ -1 \\ 0 \end{pmatrix}$$

(a) Solve the above least squares problem using normal equations.
(b) Compute the reduced singular value decomposition of A, then use it to solve the above least squares problem.

8. Let A be a 3×3 orthogonal real matrix, and $\det A = -1$. Show that there is an orthogonal matrix Q such that

$$Q^{-1}AQ = \begin{pmatrix} -1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix}$$

for some $\theta \in [0, 2\pi)$.