University of Alabama System
Doctoral Program in Applied Mathematics
Joint Program Exam in Linear Algebra and Numerical Linear Algebra

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Instructions

You may take up to three and a half hours to complete the exam. *Work seven out of the eight problems.* Completeness in your answers is very important. Justify your steps by referring to theorems by name, when appropriate, or by providing a brief theorem statement. You do not need to reprove the theorems you use. An essentially complete and correct solution to one problem will gain more credit, than solutions to two problems, each of which is "half correct". Write the last four digits of your student ID number and problem number on every page.

Notation

Throughout the exam, $\mathbb{R}$ stands for the set of real numbers and $\mathbb{C}$ for the set of complex numbers.

1. Let $A \in \mathbb{R}^{n\times n}$. For any $x, y \in \mathbb{R}^n$ define

$$\langle x, y\rangle = x^t A y.$$

Show that $\langle \cdot, \cdot \rangle$ is an inner product on $\mathbb{R}^n$ if and only if A is symmetric and positive definite.

2. Let $A \in \mathbb{C}^{n\times n}$.

   (a) Show that A has rank 1 if and only if there exist nonzero vectors $x, y \in \mathbb{C}^n$ such that $A = xy^*$.

   (b) Show that if $n > 1$ then A in Part (a) satisfies $A^2 = \langle x, y\rangle A$ (where $\langle x, y\rangle$ denotes the usual inner product). Find the minimal polynomial and a Jordan canonical form for $I + A$. Justify your answer.

3. Let V be an n dimensional vector space, and $T : V \to V$ be a linear operator. Show that if T has the same matrix representation under any basis in V, then $T = \lambda I$, where λ is some scalar and I is the identity map on V.

4. For any matrix $A = (a_{ij})$, $A \in \mathbb{R}^{m\times n}$ define

$$\|A\|_F = \left(\sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij}^2 \right)^{1/2}.$$

This is the Frobenius matrix norm. Show that if $Q \in \mathbb{R}^{m\times m}$ is orthogonal, then $\|QA\|_F = \|A\|_F$. Then show that

$$\|A\|_F = (\sigma_1^2 + \cdots + \sigma_r^2)^{1/2},$$

where σ_i are singular values of A.

5. Let $A \in \mathbb{R}^{m\times n}$. Show that

   (a) $\ker A = \ker A^t A$;

   (b) $\operatorname{rank} A^t A = \operatorname{rank} AA^t = \operatorname{rank} A$;

   (c) $A^t A$ and AA^t have the same nonzero eigenvalues. *Hint*: Keep in mind the Singular Value Decomposition of matrices.

6. Consider the linear least squares (LS) problem

$$\min_x \|b - Ax\|_2, \qquad A = \begin{bmatrix} 3 \\ 0 \\ 4 \end{bmatrix}, \quad b = \begin{bmatrix} 10 \\ 5 \\ 5 \end{bmatrix}. \tag{1}$$

   (a) Solve the LS problem (1) using the normal equation method.

   (b) Compute the QR decomposition of A using the Householder reflections and then solve the LS problem (1) using the QR decomposition method.

7. Consider the linear system $Ax = b$. Let x^* be the exact solution, and x_c be some computed approximate solution. Let $e = x^* - x_c$ be the error and $r = b - Ax_c$ be the residual for x_c. Show that

$$\frac{1}{\kappa(A)} \frac{\|r\|}{\|b\|} \le \frac{\|e\|}{\|x^*\|} \le \kappa(A) \frac{\|r\|}{\|b\|}.$$

Interpret the above inequality for $\kappa(A)$ close to 1 and for $\kappa(A)$ large.

8. (a) Let $A \in \mathbb{C}^{n\times n}$ be written as the following block form:

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix},$$

where $A_{11} \in \mathbb{C}^{k\times k}$, $A_{12} \in \mathbb{C}^{k\times(n-k)}$, $A_{21} \in \mathbb{C}^{(n-k)\times k}$, $A_{22} \in \mathbb{C}^{(n-k)\times(n-k)}$. Assume that A_{11} is nonsingular and has an LU decomposition. Let $A^{(1)} = A$. After k steps of Gaussian elimination we have

$$A^{(k+1)} = \begin{pmatrix} A_{11}^{(k+1)} & A_{12}^{(k+1)} \\ 0_{(n-k)\times k} & A_{22}^{(k+1)} \end{pmatrix}.$$

Find $A_{22}^{(k+1)}$ in terms of A_{11}, A_{12}, A_{21} and A_{22}.

(b) Let $k = 1$ and A be Hermitian and positive definite in (a). Show that after one step of Gaussian elimination $A_{22}^{(2)}$ is Hermitian and positive definite.