UNIVERSITY OF ALABAMA SYSTEM

JOINT DOCTORAL PROGRAM IN APPLIED MATHEMATICS

JOINT PROGRAM EXAMINATION

Linear Algebra and Numerical Linear Algebra

Time: Three and One Half Hours

September 12, 2002

Instructions: Do 7 of the 8 problems for full credit. Be sure to indicate which 7 are to be graded. Include all work for full credit. Write your social security number on each of your answer sheet.

[1] A matrix $A \in \mathbb{C}^{n\times n}$ is said to be *skew Hermitian* if $A^* = -A$.
(a) Prove that if A is skew Hermitian and B is unitary equivalent to A, then B is also skew Hermitian.
(b) What special form does the Shur decomposition theorem take for a skew Hermitian matrix A?
(c) Prove that the eigenvalues of a skew Hermitian matrix are purely imaginary, i.e. they satisfy $\bar{\lambda} = -\lambda$.

[2] (a) Let A be a 13×13 complex matrix with characteristic polynomial $C_A(x) = x^7(x-i)^6$, minimal polynomial $M_A(x) = x^4(x-i)^3$, and $\dim E_0 = 3$, $\dim E_i = 2$, where E_λ is the eigenspace corresponding to an eigenvalue λ of A. Find a Jordan canonical form of A.
(b) Let A be a 6×6 complex matrix with $C_A(x) = (x^2+1)^3$, $\dim E_i = 2$ and $\dim E_{-i} = 1$. Find the minimal polynomial of A.

[3] (a) Define the condition number, $\kappa(A)$, for a nonsingular matrix $A \in \mathbb{R}^{n\times n}$, show that $\kappa(A) \geq 1$ and that $\kappa(AB) \leq \kappa(A)\kappa(B)$.

(b) Consider the linear system $A\mathbf{x} = \mathbf{b}$. Let $\mathbf{x}^*$ be the exact solution, and let $\mathbf{x}_c$ be some computed approximate solution. Let $\mathbf{e} = \mathbf{x}^* - \mathbf{x}_c$ be the error and $\mathbf{r} = \mathbf{b} - A\mathbf{x}_c$ be the residual for $\mathbf{x}_c$. Show that

$$\left(\frac{1}{\kappa(A)}\right)\frac{\|\mathbf{r}\|}{\|\mathbf{b}\|} \leq \frac{\|\mathbf{e}\|}{\|\mathbf{x}^*\|} \leq \kappa(A)\frac{\|\mathbf{r}\|}{\|\mathbf{b}\|}.$$

(c) Interpret the above inequality for $\kappa(A)$ close to 1 and for $\kappa(A)$ large.

[4] Let $A \in \mathbb{C}^{n\times n}$ have two distinct eigenvalues λ_1 and λ_2. Prove that the following three statements are equivalent:
(a) A is diagonalizable,

(b) each column vector of $A - \lambda_2 I$ is in the eigenspace E_{λ_1},

(c) each column vector of $A - \lambda_2 I$ is in the eigenspace E_{λ_1} and each column vector of $A - \lambda_1 I$ is in the eigenspace E_{λ_2}.

[5] Let A be a given $n \times n$ nonsingular matrix, and assume a splitting of the form $A = M - N$, where M is nonsingular. Let $\mathbf{x}$ be the solution of the problem $A\mathbf{x} = \mathbf{b}$. Consider the iteration

$$M\mathbf{x}^{(k+1)} = \mathbf{b} + N\mathbf{x}^{(k)}.$$

Show that the errors $\mathbf{e}^{(k)} = \mathbf{x} - \mathbf{x}^{(k)}$ satisfy a relation of the form:

$$\mathbf{e}^{(k+1)} = G\mathbf{e}^{(k)}$$

and that the residuals $\mathbf{r}^{(k)} = \mathbf{b} - A\mathbf{x}^{(k)}$ satisfy a relation of the form

$$\mathbf{r}^{(k+1)} = H\mathbf{r}^{(k)}$$

for appropriate matrices G and H. How are G and H related? Prove that $\rho(H) < 1$ if and only if $\rho(G) < 1$, where $\rho(A)$ is the spectral radius of the matrix A.

[6] Let $\mathbf{u} \in \mathbb{R}^n$ be a given vector and

$$P = I - \frac{2}{\mathbf{u}^T\mathbf{u}}\mathbf{u}\mathbf{u}^T$$

be a Householder reflector matrix.
(a) Prove that P is orthogonal.

(b) Let $\mathbf{x}$ be given and let $\mathbf{x} = \mathbf{v} + \mathbf{w}$ where $\mathbf{v}$ lies along the vector $\mathbf{u}$ and $\mathbf{w}$ is orthogonal to $\mathbf{u}$. Show that $P\mathbf{x} = -\mathbf{v} + \mathbf{w}$, and explain why P is called a "reflector" matrix.

(c) For a given matrix A, explain briefly how to use Householder matrices to compute the decomposition $A = QR$ where Q is orthogonal and R is upper triangular.

[7] Consider the 3 vectors

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ \epsilon \\ 0 \\ 0 \end{pmatrix} \quad , \quad \mathbf{v}_2 = \begin{pmatrix} 1 \\ 0 \\ \epsilon \\ 0 \end{pmatrix} \quad , \quad \mathbf{v}_3 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ \epsilon \end{pmatrix}.$$

where $\epsilon \ll 1$.
(a) Use the Classical Gram-Schmidt method to compute 3 **orthonormal** vectors $\mathbf{q_1}$, $\mathbf{q_2}$ and $\mathbf{q_3}$, making the approximation that $1 + \epsilon^2 \approx 1$ (that is replace any term containing ϵ^2 or smaller with zero, **but** retain terms containing ϵ). Are all the $\mathbf{q_i}$ $(i = 1, 2, 3)$ pairwise orthogonal? If not, why not?
(b) Repeat (a) using the modified Gram-Schmidt orthogonalization process. Are the $\mathbf{q_i}$ $(i = 1, 2, 3)$ pairwise orthogonal? If not, why not?

[8] Consider the matrix

$$A = \begin{pmatrix} -2 & 11 \\ -10 & 5 \end{pmatrix}.$$

(a) Determine, a real SVD of A in the form $A = U\Sigma V^T$.

(b) What are the 1-, 2-, ∞-, and Frobenius norms of A?

(c) Find A^{-1} not directly, but via the SVD.

(d) Find the eigenvalues λ_1, λ_2 of A.

(e) Verify that det $A = \lambda_1\lambda_2$ and $|detA| = \sigma_1\sigma_2$, where σ_1 and σ_2 are singular values of A.