UNIVERSITY OF ALABAMA SYSTEM
JOINT DOCTORAL PROGRAM IN
APPLIED MATHEMATICS
JOINT PROGRAM EXAMINATION
Linear Algebra & Numerical Linear Algebra

TIME: THREE AND ONE HALF HOURS

September, 2001

Instructions: Completeness in answers is very important. Justify your steps by referring to theorems by name where appropriate. Include all work. Full credit will accrue from answering 6 of the 7 problems given. Indicate which solutions you want to be graded if you work on more than 6 problems.

1. Let ℓ^∞ be the complex vector space of all bounded sequences. Show that the functions S_+ and S_- from $\ell^\infty \to \ell^\infty$ defined by

$$(S_+ f)(1) = 0, \quad (S_+ f)(n) = f(n-1) \text{ for } n > 1$$
$$(S_- f)(n) = f(n+1) \text{ for all } n \in \mathbb{N}$$

are linear and determine all their eigenvalues and associated eigenvectors.

2. Give the definition of the term Householder reflector and find its eigenvalues, its determinant, and its singular values.

3. Consider the matrix
$$A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$$
and perform the following tasks.

   (a) Compute the eigenvalues and eigenvectors of A.

   (b) Let $x_0 = (1,0)^{\mathrm{T}}$. Compute the quantities $y_n = Ax_{n-1}$,
   $$\sigma_n = \begin{cases} y_{n;1} & \text{if } |y_{n;1}| \geq |y_{n;2}| \\ y_{n;2} & \text{if } |y_{n;1}| < |y_{n;2}| \end{cases},$$
   $x_n = y_n/\sigma_n$, and $\lambda_n = x_n^{\mathrm{T}} A x_n / (x_n^{\mathrm{T}} x_n)$ for $n = 1, ..., 4$.

   (c) What does the above algorithm apparently compute?

   (d) State and prove a theorem which explains this phenomenon.

4. True or false? For each of the following statements prove the truth or demonstrate the falsity by a counter-example, where $A, X \in \mathbb{R}^{n \times n}$, $x, b \in \mathbb{R}^n$.

   (a) $x^T A x = x^T (\frac{A + A^T}{2}) x$.

   (b) $A^k = 0$ for some positive integer k implies $A = 0$.

   (c) If A is orthogonal then $Ax = b$ can be solved in $O(n^2)$ flops.

   (d) If A is symmetric positive definite, and X is nonsingular, then $X^T A X$ is symmetric positive definite.

   (e) $\|I\|_2 = \|I\|_F = 1$, where $\|\cdot\|_F$ indicates the Frobenius norm.

5. Suppose that $A \in \mathbb{R}^{m\times n}$ and $m \geq n$. Let $A = U\Sigma V^T$ be the SVD of A, $U \in \mathbb{R}^{m\times n}$, $V \in \mathbb{R}^{n\times n}$, $\Sigma =$diag$(\sigma_1, ..., \sigma_n)$.

   (a) Determine $\|A\|_2$ using the SVD of A.

   (b) Determine an eigendecomposition of $A^T A$ in terms of the SVD of A.

   (c) Determine an eigendecomposition of AA^T in terms of the SVD of A.

   (d) Let $A = \begin{bmatrix} 3 \\ 0 \\ 4 \end{bmatrix}$.

      (i) Find the singular values and singular vectors of A. (Hint: use (b) and (c)).

      (ii) Express A as the SVD form $A = U\Sigma V^T$, where U and V are (square) orthogonal matrices and Σ is a "diagonal" matrix.

6. Consider perturbation on solving a nonsingular and real linear system $Ax = b$ of order n.

   (a) Let $(A + \delta A)y = b + \delta b$ with $\frac{\|\delta A\|}{\|A\|} \leq \epsilon$ and $\frac{\|\delta b\|}{\|b\|} \leq \epsilon$. Prove that if $\kappa(A)\epsilon = r < 1$, then $A + \delta A$ is nonsingular and $\frac{\|y\|}{\|x\|} \leq \frac{1+r}{1-r}$.

   (b) Suppose the conditions in (a) hold, prove that

$$\frac{\|x - y\|}{\|x\|} \leq \frac{2\epsilon}{1 - r}\kappa(A).$$

7. Let $A_n(c)$ be the $n \times n$ matrix defined by

$$A = \begin{bmatrix} 1 & 1 & \dots & 1 \\ \vdots & \vdots & & \vdots \\ 1 & 1 & \dots & 1 \\ c & c & c & c \end{bmatrix}$$

   (all entries are 1 except that the entries of the last row are equal to c.) For every positive integer $n \geq 2$ and $c \in \mathbb{C}$, determine a Jordan canonical form $J_n(c)$ that is similar to $A_n(c)$.