UNIVERSITY OF ALABAMA SYSTEM
JOINT DOCTORAL PROGRAM IN APPLIED MATHEMATICS
JOINT PROGRAM EXAMINATION
Linear Algebra and Numerical Linear Algebra

TIME: THREE AND ONE HALF HOURS

May, 2001

Instructions: Do 7 of the 8 problems for full credit. Be sure to indicate which 7 are to be graded. Include all work. Write your student ID number on every page of your exam.

1. Let V be the vector space of polynomials of degree at most 2 with complex coefficients and consider the linear transformation $D : V \to V$, $y \mapsto y'$. Find the eigenvalues of D, their geometric and algebraic multiplicities, and the minimal and characteristic polynomials of D. Determine a basis of V such that the matrix of D with respect to this basis is in Jordan canonical form.

2. Let $A \in \mathbb{R}^{n\times n}$ be given, singular. Use the Schur Theorem to show that, for any $\epsilon > 0$, there is a non-singular matrix A_ϵ such that $\|A - A_\epsilon\|_2 \leq \epsilon$. Can a similar statement be proved for an arbitrary defective matrix A and a non-defective matrix A_ϵ?

3. Suppose $A \in \mathbb{C}^{m\times n}$ has rank n and $b \in \mathbb{C}^n$. Prove that the block linear system

$$\begin{bmatrix} I_{m\times m} & A \\ A^* & 0_{n\times n} \end{bmatrix} \begin{bmatrix} r \\ x \end{bmatrix} = \begin{bmatrix} b \\ 0_{n\times n} \end{bmatrix}$$

has a unique solution $(r, x)^T$ where $r \in \mathbb{C}^m$ and $x \in \mathbb{C}^n$. Show that r and x must be the residual and solution of the least squares problem for minimizing $\|b - Ax\|_2$.

4. Let A and B be two linear transformations such that $AB - BA = I$, the identity. Show that $A^kB - BA^k = kA^{k-1}$, for all integers $k > 1$.

5. Given a non-singular $A \in \mathbb{R}^{n\times n}$, show that
(a) AA^T and A^TA have the same eigenvalues, all positive, but (generally) different eigenvectors,
(b) if these eigenvalues are arranged in descending order of magnitude, the condition number $\kappa_2(A) = \sqrt{\lambda_1/\lambda_n}$,
(c) the condition

$$\frac{\| \delta A \|}{\| A \|} < \frac{1}{\kappa(A)},$$

for any norm, guarantees that the perturbed matrix $(A + \delta A)$ is non-singular.

6. (a) Let V be a finite-dimensional subspace of $\mathbb{C}^n$. Prove that for any $x \in \mathbb{C}^n$, there exists $p \in V$ and $q \in \mathbb{C}^n$ such that $x = p + q$ and $(y, q) = 0$ for all $y \in V$. (b) Let $\mathcal{V}$ be an inner product space and $\mathcal{W}$ a finite dimensional subspace of $\mathcal{V}$. For $x \in \mathcal{V}$, show that the orthogonal projection of x onto $\mathcal{W}$ is the unique vector in $\mathcal{W}$ closest to x.

7. Let $A \in \mathbb{R}^{n \times n}$ be a symmetric matrix with eigenvalues such that $|\lambda_1| > |\lambda_2| \geq |\lambda_3| \geq \ldots \geq |\lambda_n| > 0$. Suppose $z \in \mathbb{R}^n$ with $z^T x_1 \neq 0$, where $Ax_1 = \lambda_1 x_1$. Prove that, for some constant C,

$$\lim_{k \to \infty} \frac{A^k z}{\lambda_1^k} = C x_1$$

and describe a reliable algorithm, based on this result, for computing λ_1 and x_1. Explain how the calculation should be modified to obtain (a) λ_n and (b) the eigenvalue closest to 2.

8. Let $T : V \to W$, $U : W \to V$ be linear transformations such that $(UT)(x) = x, \forall x \in V$ where $\dim V = \dim W < \infty$. Without assuming invertibility, establish the following:

   (a) T is $1 - 1$;

   (b) T is onto;

   (c) T^{-1} exists and $T^{-1} = U$;

   (d) If A and B are square matrices with $AB = I$, then both A and B are invertible and $A^{-1} = B, B^{-1} = A$.