UNIVERSITY OF ALABAMA SYSTEM
JOINT DOCTORAL PROGRAM IN APPLIED MATHEMATICS
JOINT PROGRAM EXAMINATION
Linear Algebra and Numerical Linear Algebra

TIME: THREE AND ONE HALF HOURS

May, 2000

Instructions: Do 7 of the 8 problems for full credit. Be sure to indicate which 7 are to be graded. Include all work. Write your student ID number on every page of your exam.

1. Let V be a vector space of finite dimension n, let T be a linear operator on V with $k+1$ distinct eigenvalues $\lambda_0, \lambda_1, \lambda_2, \ldots, \lambda_k$, and let the eigenspace corresponding to λ_0 have dimension $n-k$. Prove that the operator T^m is diagonalizable for each positive integer m.

2. Let V and W be finite dimensional vector spaces, and let $T : V \to W$ be a linear transformation of rank r where $1 \leq r < \min\{\dim(V), \dim(W)\}$. Prove that there exist bases α for V and β for W such that the matrix representation for T with respect to α and β has the form $\begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix}$.

3. Let V be a finite dimensional vector space with inner product $\langle \cdot, \cdot \rangle$, and let T be a self-adjoint operator on V. Prove that there exists a self-adjoint operator S on V such that $T = S^2$ if and only if $\langle T\mathbf{x}, \mathbf{x} \rangle \geq 0$ for all $\mathbf{x} \in V$.

4. (a) Let A be a 10×10 complex matrix with characteristic polynomial $C_A(x) = (x-1)^6(x+2)^4$, minimal polynomial $M_A(x) = (x-1)^3(x+2)^2$, and $\dim E_1 = 3$, $\dim E_{-2} = 2$, where E_1 and E_{-2} are the eigenspaces corresponding to the eigenvalues 1 and -2 respectively. Find a Jordan canonical form of A.

   (b) Let A be an 8×8 complex matrix with characteristic polynomial $C_A(x) = (x+i)^3(x-i)^3(x-1)^2$, and $\dim E_{-i} = \dim E_i = \dim E_1 = 2$. Find the minimal polynomial of A.

5. (a) Calculate A^{-1} and $\kappa_\infty(A)$ for the matrix

$$A = \begin{bmatrix} 375 & 374 \\ 752 & 750 \end{bmatrix}.$$

   (b) For the above A, find $\mathbf{b}$, $\delta\mathbf{b}$ and $\mathbf{x}$, $\delta\mathbf{x}$ such that

$$A\mathbf{x} = \mathbf{b}, \qquad A(\mathbf{x} + \delta\mathbf{x}) = \mathbf{b} + \delta\mathbf{b}$$

   with $\| \delta\mathbf{b} \|_\infty / \| \mathbf{b} \|_\infty$ small and $\| \delta\mathbf{x} \|_\infty / \| \mathbf{x} \|_\infty$ large.

   (c) Let $A \in \mathbb{R}^{n \times n}$ be given, nonsingular, and consider the linear system problem

$$A\mathbf{x} = \mathbf{b},$$

   where $\mathbf{b} \in \mathbb{R}^n$ is given. Let $\mathbf{x} + \delta\mathbf{x} \in \mathbb{R}^n$ be an approximate solution to this problem, satisfying

$$A(\mathbf{x} + \delta\mathbf{x}) = \mathbf{b} + \delta\mathbf{b}.$$

   Prove that

$$\frac{\| \delta\mathbf{x} \|}{\| \mathbf{x} \|} \leq \kappa(A) \frac{\| \delta\mathbf{b} \|}{\| \mathbf{b} \|}$$

   and comment on the significance of this result.

6. Use a QR decomposition, with exact arithmetic, to solve the least squares problem for the overdetermined system

$$\begin{bmatrix} 1 & -3 \\ 2 & 4 \\ 2 & 5 \end{bmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{bmatrix} 4 \\ 3 \\ -5 \end{bmatrix}.$$

State the magnitude of the minimum residual.

7. Let $A \in \mathbb{R}^{n \times n}$ be given, and let Q_0 be an arbitrary $n \times n$ orthogonal matrix. Consider the sequence of matrices R_k and Q_k computed as follows:

$$\begin{aligned} Z_{k+1} &= AQ_k, \\ Q_{k+1}R_{k+1} &= Z_{k+1}. \end{aligned}$$

In the last step, we compute the QR decomposition of Z_{k+1} to get Q_{k+1} and R_{k+1}. Assume that

$$\lim_{k \to \infty} Q_k = Q_\infty$$

and

$$\lim_{k \to \infty} R_k = R_\infty$$

exist. Prove that the eigenvalues of A are given by the diagonal elements of R_∞.

8. Let $A \in \mathbb{C}^{n \times n}$ be given, Hermitian, and let (λ, u) be an arbitrary eigenpair of A, with u real and $\|u\|_2 = 1$. Let $x \approx u$ be given, with $\|x\|_2 = 1$ and define σ by

$$\sigma = \frac{(Ax, x)}{(x, x)},$$

the Rayleigh quotient of x. Prove that

$$|\lambda - \sigma| \leq C\|u - x\|_2^2.$$