# UNIVERSITY OF ALABAMA SYSTEM
# JOINT DOCTORAL PROGRAM IN APPLIED MATHEMATICS
# JOINT PROGRAM EXAMINATION
## Linear Algebra and Numerical Linear Algebra

TIME: THREE AND ONE HALF HOURS

September 16, 1999

**Instructions:** Do 7 of the 8 problems for full credit. Be sure to indicate which 7 are to be graded. Include all work. Write your student ID number on every page of your exam.

1. Consider the complex matrix

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega^2 \end{bmatrix}, \quad \text{where } \omega = \frac{-1+\sqrt{3}\,i}{2}$$

(Note that $\omega^3 = 1$.) Let $S$ be the set of all polynomials of matrix $A$ with real coefficients.

   (a) Introduce an addition and scalar multiplication to make $S$ a vector space over the field of real numbers.

   (b) For the vector space $S$ defined in (a), find its dimension and a set of basis vectors.

2. (a) Given $x = (2,2,1)^T$, find an orthogonal matrix $Q$ such that $Qx$ is parallel to $e_1 = (1,0,0)^T$.
(b) Find an orthogonal matrix $Q$ and an upper triangular matrix $R$ such that $A = QR$, where

$$A = \begin{bmatrix} 3 & 1 & 2 \\ -4 & 2 & 4 \\ 0 & 0 & 5 \end{bmatrix}$$

3. Let $A \in \mathbb{R}^{n\times m}$ be given. Let the singular value decomposition of $A$ be given by $A = USV^T$, where $U \in \mathbb{R}^{n\times n}$ and $V \in \mathbb{R}^{m\times m}$ are orthogonal, and $S \in \mathbb{R}^{n\times m}$ is zero except for the diagonal elements $s_{ii} = \sigma_i$ which are the singular values of $A$. For a given vector $b \in \mathbb{R}^n$, define $x \in \mathbb{R}^m$ by

$$x = \sum \frac{u_i^T b}{\sigma_i} v_i$$

where $u_i$ denotes the $i^{th}$ column of $U$ (and similarly for $v_i$ and $V$), and the sum is taken over the non-zero singular values of $A$. Show that

$$\|b - Ax\|_2 \leq \|b - Ay\|_2$$

for all $y \in \mathbb{R}^m$, $y \neq x$. What condition is needed to make the inequality strict?

4. Let $A \in \mathbb{R}^{4\times 4}$ be given, with spectrum

$$\sigma(A) = \{0.01, -1.25, 3.46, -10\}.$$

Find the best lower bound for

$$M(A) = \max_{x\neq 0} \frac{\|Ax\|}{\|x\|}$$

and the best upper bound for

$$m(A) = \min_{x\neq 0} \frac{\|Ax\|}{\|x\|}.$$

5. Let $A \in \mathbb{R}^{n\times n}$ satisfy $A^2 = I$, where $I$ is the identity matrix. Show that $\text{rank}(A + I) + \text{rank}(A - I) = n$.

6. Let $A \in \mathbb{R}^{n\times n}$ be symmetric positive definite. Show that

$$\det \begin{bmatrix} a_{11} & \cdots & a_{1n} & x_1 \\ & \cdots & & \vdots \\ a_{n1} & \cdots & a_{nn} & x_n \\ x_1 & \cdots & x_n & 0 \end{bmatrix} < 0$$

for every non-zero vector $x = (x_1, \ldots, x_n)$.

7. Given $A \in \mathbb{R}^{n\times n}$, consider the following iteration:

Given $x_0 \in \mathbb{R}^n$ compute as follows:

(a) $z_k = Ax_k$;

(b) $\sigma_k = z_{k,i}$, where $\|z_k\|_\infty = |z_{k,i}|$ (here $z_{k,i}$ is the $i^{th}$ component of the vector $z_k$);

(c) $x_{k+1} = z_k/\sigma_k$.

State and prove a theorem showing when (and to what) this iteration converges.

8. Let $A \in \mathbb{C}^{n\times n}$. We say that $A$ has a square root, if there exists a matrix $B \in \mathbb{C}^{n\times n}$ such that $A = B^2$.

(a) Let $A$ be similar to a matrix $J$ (i.e. $A = P^{-1}JP$ for some $P \in \mathbb{C}^{n\times n}$). Show that $A$ has a square root if and only if $J$ also has a square root.

(b) Let $J = \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}$ be a $2 \times 2$ Jordan block. Show that $J$ has a square root if and only if $\lambda \neq 0$.