UNIVERSITY OF ALABAMA SYSTEM
JOINT DOCTORAL PROGRAM IN APPLIED MATHEMATICS
JOINT PROGRAM EXAMINATION
Linear Algebra and Numerical Linear Algebra

TIME: THREE AND ONE HALF HOURS

May, 1999

Instructions: Do 7 of the 8 problems for full credit. Include all work.

1. (a) Define the condition number, $\kappa(A)$, for a nonsingular matrix $A \in \mathbb{R}^{n\times n}$; show that $\kappa(A) \geq 1$ and that $\kappa(AB) \leq \kappa(A)\kappa(B)$.

   (b) Consider the linear system $Ax = b$. Let x_* be the exact solution, and let x_c be some computed approximate solution. Let $e = x_* - x_c$ be the error and $r = b - Ax_c$ be the residual for x_c. Show that

$$\left(\frac{1}{\kappa(A)}\right)\frac{\|r\|}{\|b\|} \leq \frac{\|e\|}{\|x_*\|} \leq \kappa(A)\frac{\|r\|}{\|b\|}.$$

   (c) Interpret the above inequality for $\kappa(A)$ close to 1 and for $\kappa(A)$ large.

2. (a) Let $\sigma_1, \ldots, \sigma_r$ be the non-zero singular values of a matrix $A \in \mathbb{R}^{m\times n}$. Show that $\sigma_1^2, \ldots, \sigma_r^2$ are the non-zero eigenvalues of both A^TA and AA^T.

   (b) Let $A \in \mathbb{R}^{n\times n}$ be non-singular. Show that

$$\kappa_2(A) = \frac{\sigma_1}{\sigma_n},$$

   where $\kappa_2(A)$ is the 2-condition number of A, σ_1 is the largest singular value of A, and σ_n is the smallest singular value of A.

3. (a) Show that the matrices $A = \begin{pmatrix} a & 0 \\ b & a \end{pmatrix}$ with $b \neq 0$ and $B = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}$ are *not* similar. Based on this, prove that the matrix A is not diagonalizable over the complex field.
(b) Find a 2×2 matrix A such that A^2 is diagonalizable but A is not.

4. Let $S : \mathbb{R}^{n\times n} \to \mathbb{R}^{n\times n}$ be the transformation defined by $S(A) = (A + A^T)/2$.

   (a) Prove that S is linear.

   (b) Find a basis of the null space of S and determine its dimension

5. A *projector* is a square matrix P that satisfies $P^2 = P$. A projector P is an *orthogonal projector* if $null(P)$ is orthogonal to $range(P)$. Let $P \in \mathbb{C}^{n\times n}$ be a nonzero projector. Show that $\|P\|_2 \geq 1$, with equality if and only if P is an orthogonal projector.

6. Let $T : V \to W$, $U : W \to V$ be linear transformations such that $(UT)(x) = x, \forall x \in V$ where $\dim V = \dim W < \infty$. Without assuming invertibility, establish the following:

   (a) T is $1-1$;

   (b) T is onto;

(c) T^{-1} exists and $T^{-1} = U$;

(d) If A and B are square matrices with $AB = I$, then both A and B are invertible and $A^{-1} = B, B^{-1} = A$.

7. Let $A \in \mathbb{C}^{8\times 8}$ have characteristic polynomial $C_A(x) = (x-3)^8$, minimal polynomial $M_A(x) = (x-3)^4$ and $\dim E_3 = 3$, where E_3 is the eigenspace of A corresponding to the eigenvalue 3. List all the possible Jordan canonical forms for A and give reasons for your answer.

8. The matrix

$$A = \begin{bmatrix} \beta & 0 & 0 \\ 0 & 1 & 4 \\ 0 & 4 & 1 \end{bmatrix}$$

has eigenpairs

$$(\lambda, x) = \left(\beta, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right), \left(-3, \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}\right), \left(5, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}\right)$$

Assume $3 < \beta < 5$ and suppose the power method is applied with starting vector

$$z_0 = \alpha[1, 1, -1]^T$$

where $0 < \alpha \leq 1$.

(a) Determine whether or not the iteration will converge to an eigenpair of A, and if so, which one. Assume exact arithmetic.

(b) Repeat (a), except we now use inverse iteration using the same starting vector z_0 and the Rayleigh quotient of z_0 as approximation for the eigenvalue.

(c) If we did the calculations for (a) and (b) in standard floating point arithmetic, what should we expect to happen, for most values of β and α?