UNIVERSITY OF ALABAMA SYSTEM
JOINT DOCTORAL PROGRAM IN APPLIED MATHEMATICS
JOINT PROGRAM EXAMINATION
Numerical Linear Algebra

TIME: THREE AND ONE HALF HOURS

September, 1998

Instructions: Completeness in answers is very important. Justify your steps by referring to theorems by name where appropriate. Include all work. Full credit will accrue from answering 7 of the 8 problems given. Indicate which solutions you want to be graded if you work on more than 7 problems.

1. Let $A \in \mathbf{R}^{n\times n}$ for $n = 2$ be orthogonal with $\det A = -1$. Show that there exists $\theta \in [0, 2\pi)$ such that

$$A = \begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix},$$

Show also that 1 and -1 are eigenvalues of the matrix A, and the corresponding eigenvectors are orthogonal.

2. Let $A \in \mathbf{R}^{n\times m}$ and $B \in \mathbf{R}^{m\times n}$ with $n \geq m$. Assume that $\operatorname{rank}(A) = m$, $(AB)^T = AB$ and $ABA = A$. Show that $B = (A^TA)^{-1}A^T$.

3. (a) Let A be a 16×16 complex matrix whose characteristic and minimal polynomials are $C_A(x) = x^{10}(x-3i)^6$ and $M_A(x) = x^6(x-3i)^3$, respectively. Also let $\dim E_0 = 2$, $\dim E_{3i} = 3$, where E_λ is an eigenspace corresponding to the eigenvalue λ of A. Find a Jordan canonical form of A.

   (b) Let A be a 10×10 complex matrix with $C_A(x) = (x^2+1)^5$, $\dim E_i = 1$ and $\dim E_{-i} = 4$. Find the minimal polynomial of A.

4. For both these problems it might be good to recall the Schur Decomposition: For any $A \in \mathbf{C}^{n\times n}$, there exists a unitary $U \in \mathbf{C}^{n\times n}$ and a triangular $T \in \mathbf{C}^{n\times n}$ such that $A = U^HTU$.

   (a) Let $A \in \mathbf{C}^{n\times n}$ be given, singular. Show that, for any $\epsilon > 0$, there exists a nonsingular matrix $A_\epsilon \in \mathbf{C}^{n\times n}$, such that

   $$\|A_\epsilon - A\|_2 \leq \epsilon.$$

   (b) Let $A \in \mathbf{C}^{n\times n}$ be given, defective. Show that, for any $\epsilon > 0$, there exists a diagonalizeable matrix $A_\epsilon \in \mathbf{C}^{n\times n}$, such that

   $$\|A_\epsilon - A\|_2 \leq \epsilon.$$

   Comment on the significance of both of these results.

5. Let V be a finite-dimensional inner product space, and let W be a subspace of V. Then $V = W \oplus W^\perp$, that is, each $\alpha \in V$ is uniquely expressed in the form $\alpha = \beta + \gamma$ with $\beta \in W$ and $\gamma \in W^\perp$. Define a linear operator U by $U\alpha = \beta - \gamma$.

   (a) Prove that U is both self-adjoint and unitary.

   (b) If V is $\mathbf{R}^3$ with the standard inner product and W is the subspace spanned by $[1,0,1]^T$, find the matrix representation of U in the standard ordered basis (i.e., $e_1 = (1,0,0)^T$, $e_2 = (0,1,0)^T$, and $e_3 = (0,0,1)^T$).

6. Let $A \in \mathbf{R}^{n\times n}$ be given, symmetric, and assume that the eigenvalues of A satisfy

   $$|\lambda_1| > |\lambda_2| \geq \ldots \geq |\lambda_{n-1}| \geq |\lambda_n|.$$

   Let $z \in \mathbf{R}^n$ be given. Under what conditions on z does the following hold, theoretically? (Be sure to actually show that it holds!)

   $$\lim_{k\to\infty} \frac{z^TA^{k+1}z}{z^TA^kz} = \lambda_1$$

   Under what conditions on z does this hold, *as a practical matter?* Explain fully for full credit.

7. Show that $A \in \mathbf{C}^{n\times n}$ is nilpotent (i.e., $A^k = 0$ for some positive integer k) if and only if all eigenvalues of A are zero. Show that if A is nilpotent, then $A + I$ is nonsingular.

8. (a) Define the condition number, $\kappa(A)$, for a nonsingular matrix $A \in \mathbf{R}^{n\times n}$; show that $\kappa(A) \geq 1$ and that $\kappa(AB) \leq \kappa(A)\kappa(B)$.

(b) Consider the linear system $Ax = b$. Let x_* be the exact solution, and let x_c be some computed approximate solution. Let $e = x_* - x_c$ be the error and $r = b - Ax_c$ be the residual for x_c. Show that

$$\left(\frac{1}{\kappa(A)}\right)\frac{\|r\|}{\|b\|} \leq \frac{\|e\|}{\|x_*\|} \leq \kappa(A)\frac{\|r\|}{\|b\|}.$$

(c) Interpret the above inequality for $\kappa(A)$ close to 1 and for $\kappa(A)$ large.