UNIVERSITY OF ALABAMA SYSTEM
JOINT DOCTORAL PROGRAM IN APPLIED MATHEMATICS
JOINT PROGRAM EXAMINATION
Linear Algebra / Numerical Linear Algebra

TIME: THREE AND ONE HALF HOURS

September, 1997

Instructions: Completeness in answers is very important. Justify your steps by referring to theorems by name where appropriate. Include all work. Full credit will be given for correctly answering 6 of the 7 problems given. Indicate which solutions you want to be graded if you work on more than 6 problems.

1. Let $V = \mathbf{R}^{n\times n}$ be the vector space of all real $n \times n$ matrices and $T : V \to V$ be the transformation defined by $T(A) = \frac{1}{2}(A + A^T)$.

   (a) Prove that T is linear.

   (b) Find a basis of the null space of T and determine its dimension.

   (c) Find a basis of the range of T and determine its dimension.

2. Let $\mathcal{P}_3(\mathbf{R})$ denote the space of all polynomials of degree ≤ 3 with real coefficients. Find $p \in \mathcal{P}_3(\mathbf{R})$ such that $p(0) = 0$ and

$$\int_{-1}^{1} (2 + 3t - p(t))^2 \, dt$$

   is as small as possible.

3. (a) Find a Jordan form J for $A = \begin{pmatrix} i & 1 \\ 1 & -i \end{pmatrix}$ and a nonsingular P such that $P^{-1}AP = J$

   (b) Prove that every complex 2×2 matrix is similar to a symmetric matrix.

4. Let V be a finite dimensional inner product space with an inner product $< \cdot, \cdot >$ and a norm $\| \cdot \| = \sqrt{< \cdot, \cdot >}$. Let a linear operator $T : V \to V$ be self–adjoint. The Rayleigh quotient for $x \neq 0$ is defined as

$$R(x) = \frac{< T(x), x >}{\|x\|^2}.$$

   Prove that $\max_{x \neq 0} R(x)$ is the largest eigenvalue of T and $\min_{x \neq 0} R(x)$ is the smallest eigenvalue of T.

5. Consider the iteration: $Q_{k+1} R_{k+1} = A Q_k$, where $A \in \mathbf{C}^{n\times n}$ is nonsingular, $Q_0 = I$, $Q_k \in \mathbf{C}^{n\times n}$ is unitary, and $R_k \in \mathbf{C}^{n\times n}$ is upper triangular. Prove that there exists an upper triangular matrix U_k such that $Q_k = A^k U_k$ and a lower triangular matrix L_k such that $Q_k = (A^H)^{-k} L_k$, where A^H is the conjugate transpose of A.

6. Let

$$A = \begin{pmatrix} 3 & 3 \\ 0 & 4 \\ 4 & -1 \end{pmatrix} \qquad \text{and} \qquad b = \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}.$$

(a) Use the Gram–Schmidt process to find an orthonormal basis for the column space of A.

(b) Factor A into a product QR, where $Q \in \mathbf{R}^{3\times 2}$ has an orthonormal set of column vectors and $R \in \mathbf{R}^{2\times 2}$ is upper triangular.

(c) Solve the least squares problem $Ax = b$.

7. (a) Show that given an invertible matrix $A \in \mathbf{R}^{n\times n}$, one can choose vectors $b \in \mathbf{R}^n$ and $\Delta b \in \mathbf{R}^n$ so that if

$$\begin{aligned} Ax &= b, \\ A(x + \Delta x) &= b + \Delta b, \end{aligned}$$

then

$$\frac{\|\Delta x\|_2}{\|x\|_2} = \kappa_2(A)\frac{\|\Delta b\|_2}{\|b\|_2},$$

where $\kappa_2(A) = \|A\|_2 \|A^{-1}\|_2$ is the 2−condition number.

(b) Explain the significance of part (a) for the 'optimal' role of condition numbers in the sensitivity analysis of linear systems.