UNIVERSITY OF ALABAMA SYSTEM
JOINT DOCTORAL PROGRAM IN APPLIED MATHEMATICS
JOINT PROGRAM EXAMINATION
Linear Algebra / Numerical Linear Algebra

TIME: THREE AND ONE HALF HOURS

May, 1996

Instructions: Completeness in answers is very important. Justify your steps by referring to theorems by name where appropriate. An essentially complete and correct solution to one problem will gain more credit than solutions to two problems, each of which is "half-correct". Full credit will be given for correctly answering 5 of the 7 problems given.

1. Let V be the real vector space of functions on the real line spanned by the basis $B = \{\sin x, \cos x, x\sin x, x\cos x\}$. Let T be defined by $Tf = f''$, where f'' denote the second derivative of f with respect to x.

   (a) Prove that T is a linear operator on V.

   (b) Find the matrix $\mathbf{A}$ which represents T with respect to the basis B.

   (c) Determine the characteristic polynomial, the eigenvalues, the dimensions of eigenspaces, the minimal polynomial, and the Jordan canonical form of $\mathbf{A}$.

2. Let T be a linear operator on a finite dimensional complex inner product space V, and let T^* be the adjoint of T. Prove that $T = T^*$ if and only if $T^*T = T^2$.

3. Let V be a finite dimensional complex inner product space, W a subspace of V, and
$$W^{\perp} = \{\mathbf{v} \in V :< \mathbf{v}, \mathbf{w} >= 0 \text{ for all } \mathbf{w} \in W\}$$
the orthogonal complement of W in V.

   (a) Show that $W^{\perp}$ is a subspace of V and
$$V = W \oplus W^{\perp}.$$

   (b) Prove that $(W^{\perp})^{\perp} = W$.

   (c) In the vector space of real polynomials of degree at most 2 with the inner product
$$< f, g >= \int_0^1 f(x)g(x)dx,$$
compute $(\text{span}\{1, x\})^{\perp}$.

4. Let $\mathbf{A} \in \mathbb{R}^{m\times n}$ with $m > n$.

   (a) Define the term "the least squares solution" for $\mathbf{Ax} = \mathbf{b}$.

   (b) Derive the normal equation of the least squares problem associated with $\mathbf{Ax} = \mathbf{b}$

   (c) Prove that the least squares problem associated with $\mathbf{Ax} = \mathbf{b}$ has a unique solution if and only if $\text{rank}(\mathbf{A}) = n$.

5. Let $\mathbf{w}$ be a unit vector in $\mathbb{R}^n$. Prove that:

   (a) The matrix $\mathbf{H} = \mathbf{I} - 2\mathbf{w}\mathbf{w}^T$ is symmetric and orthogonal.

   (b) $\mathbf{Hw} = -\mathbf{w}$.

   (c) If $\mathbf{u}$ is orthogonal to $\mathbf{w}$, then $\mathbf{Hu} = \mathbf{u}$.

   (d) Explain why $\mathbf{H}$ can be interpreted as a reflection to the subspace
$$(\text{span}\{\mathbf{w}\})^{\perp} = \{\mathbf{u} : \mathbf{w}^T\mathbf{u} = 0\}.$$

6. (a) Define the condition number κ of a matrix $\mathbf{A} \in \mathbb{R}^{n\times n}$.

   (b) Show that the 2-condition number $\kappa_2(\mathbf{A})$ equals 1 if and only if $\mathbf{A}$ is a constant multiple of an orthogonal matrix.

   (c) Compute $\kappa_1(\mathbf{A})$, $\kappa_{\mathbf{2}}(\mathbf{A})$, $\kappa_\infty(\mathbf{A})$ for

$$A = \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}$$

   (d) State a formula to express the effect of the condition number of $\mathbf{A}$ on the stability of solutions of $\mathbf{Ax} = \mathbf{b}$ under perturbations of $\mathbf{A}$ and $\mathbf{b}$. For which condition numbers $\kappa(\mathbf{A})$ can the numerical solution of $\mathbf{Ax} = \mathbf{b}$ on a machine with unit round off u not be trusted?

7. Let $\mathbf{A} \in \mathbb{R}^{m\times n}$ with $m \geq n$ and $\operatorname{rank}(\mathbf{A}) = n$. Given the $\mathbf{QR}$ factorization of $\mathbf{A}$ with the orthogonal $\mathbf{Q} \in \mathbb{R}^{m\times m}$ and upper triangular $\mathbf{R} \in \mathbb{R}^{m\times n}$, prove that

   (a) If $\mathbf{A} = \begin{pmatrix} \mathbf{a}_1 \dots \mathbf{a}_n \end{pmatrix}$ and $\mathbf{Q} = \begin{pmatrix} \mathbf{q}_1 \dots \mathbf{q}_m \end{pmatrix}$ are column partitionings, then

$$\operatorname{span}\{\mathbf{a}_1, \dots, \mathbf{a}_k\} = \operatorname{span}\{\mathbf{q}_1, \dots, \mathbf{q}_k\} \quad \text{for} \quad k = 1. \dots, n.$$

   In particular, if $\mathbf{Q}_1 = \begin{pmatrix} \mathbf{q}_1 \dots \mathbf{q}_n \end{pmatrix}$ and $\mathbf{Q}_2 = \begin{pmatrix} \mathbf{q}_{n+1} \dots \mathbf{q}_m \end{pmatrix}$ then

$$\operatorname{range}(\mathbf{A}) = \operatorname{range}(\mathbf{Q}_1) \quad \text{and} \quad (\operatorname{range}(\mathbf{A}))^\perp = \operatorname{range}(\mathbf{Q}_2),$$

   and

$$\mathbf{A} = \mathbf{Q}_1\mathbf{R}_1 \quad \text{with} \quad \mathbf{R}_1 = \begin{pmatrix} \mathbf{r}^1 \\ \vdots \\ \mathbf{r}^n \end{pmatrix},$$

   where $\mathbf{R} = \begin{pmatrix} \mathbf{r}^1 \\ \vdots \\ \mathbf{r}^m \end{pmatrix}$ is row partitioning of $\mathbf{R}$.

   (b) The above introduced "skinny" $\mathbf{QR}$ factorization $\mathbf{A} = \mathbf{Q}_1\mathbf{R}_1$, where $\mathbf{Q}_1 \in \mathbb{R}^{m\times n}$ has orthonormal columns and $\mathbf{R}_1 \in \mathbb{R}^{n\times n}$ is upper triangular, can be chosen such that $\mathbf{R}_1$ has positive diagonal entries. Under this additional requirement $\mathbf{Q}_1$ and $\mathbf{R}_1$ become unique and $\mathbf{R}_1 = \mathbf{G}^T$ where $\mathbf{G}$ is the lower triangular Cholesky factor of $\mathbf{A}^T\mathbf{A}$.