COMPLEX ANALYSIS

Lecture notes for
MA 648

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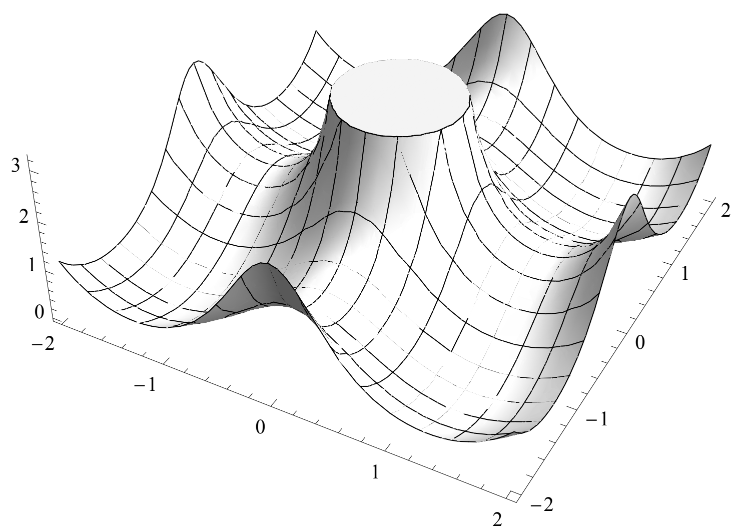

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Preface

Complex Analysis, also called the Theory of Functions, is one of the most important and certainly one of the most beautiful branches of mathematics. This is due to the fact that, in the case of complex variables, differentiability in open sets has consequences which are much more significant than in the case of real variables. It is the goal of this course to study these consequences and some of their far reaching applications.

As a prerequisite of the course familiarity with Advanced Calculus[1] is necessary and, generally, sufficient. Some prior exposure to topology may be useful but the necessary concepts and theorems are provided in the appendix if they are not covered in the course. In order to help you navigate the notes an index, a list of symbols, and a glossary of important terms (not explicitly defined in the text) are appended at the end.

There are many textbooks on our subject; several of the following have been consulted in the preparation of these notes.

- Lars V. Ahlfors, *Complex analysis*, McGraw-Hill, New York, 1953.
- John B. Conway, *Functions of one complex variable*, Springer, New York, 1978.
- Walter Rudin, *Real and complex analysis*, McGraw-Hill, New York, 1987.
- Elias M. Stein and Rami Shakarchi, *Complex analysis*, Princeton University Press, Princeton, 2003.

These lecture notes are not intended to be encyclopedic; I tried, rather, to be pedagogic. As a consequence there are many occasions where assumptions could be weakened or conclusions strengthened. Also, very many interesting results living close by have been skipped and other interesting and fruitful subjects have not even been touched. This is just the very beginning of the most exciting and beautiful Theory of Functions.

Thanks to my classes of Fall 2013, 2016, and 2019 who caused many improvements to the notes and found a large number of mistakes (but probably not all).

[1] Among many sources my Advanced Calculus notes would serve in this respect. They may be found at `http://people.cas.uab.edu/~weikard/teaching/ac.pdf`.

CHAPTER 1

The complex numbers: algebra, geometry, and topology

1.1. The algebra of complex numbers

1.1.1 The field of complex numbers. The set $\mathbb{C}$ of complex numbers is the set of all ordered pairs of real numbers equipped with two *binary operations* $+$ and $\cdot$ as follows: if (a, b) and (c, d) are pairs of real numbers, then

$$(a, b) + (c, d) = (a + c, b + d)$$

and

$$(a, b) \cdot (c, d) = (ac - bd, ad + bc).$$

The set $\mathbb{C}$ with the binary operations $+$ and $\cdot$ is a field, i.e., the following properties hold:

(A1): $(x + y) + z = x + (y + z)$ for all $x, y, z \in \mathbb{C}$ (*associative law of addition*).
(A2): $x + y = y + x$ for all $x, y \in \mathbb{C}$ (*commutative law of addition*).
(A3): $(0, 0)$ is the unique *additive identity*.
(A4): Each $x \in \mathbb{C}$ has a unique *additive inverse* $-x \in \mathbb{C}$, called the *negative* of x.
(M1): $(x \cdot y) \cdot z = x \cdot (y \cdot z)$ for all $x, y, z \in \mathbb{C}$ (*associative law of multiplication*).
(M2): $x \cdot y = y \cdot x$ for all $x, y \in \mathbb{C}$ (*commutative law of multiplication*).
(M3): $(1, 0)$ is the unique *multiplicative identity*.
(M4): Each $x \in \mathbb{C} \setminus \{(0, 0)\}$ has a unique *multiplicative inverse* $x^{-1} \in \mathbb{C}$, called the *reciprocal* of x.
(D): $(x + y) \cdot z = x \cdot z + y \cdot z$ for all $x, y, z \in \mathbb{C}$ (*distributive law*).

For simplicity we will usually write $x - y$ in place of $x + (-y)$, and $\frac{x}{y}$ or x/y for $x \cdot y^{-1}$. It is also common to write xy instead of $x \cdot y$ and to let multiplication take precedence over addition, i.e., $x + yz$ is short for $x + (yz)$.

1.1.2 Embedding of $\mathbb{R}$ into $\mathbb{C}$. The identification of a real number x with the complex number $(x, 0)$ is an injective map and addition and multiplication of real numbers are faithfully reproduced in $\mathbb{C}$, i.e., $(x, 0) + (y, 0) = (x + y, 0)$ and $(x, 0)(y, 0) = (xy, 0)$. Thus we may treat $\mathbb{R}$ as a subfield of $\mathbb{C}$.

Next note that $(x, y) = (x, 0) + (0, 1)(y, 0)$. Abbreviating $(0, 1)$ by i this becomes $x + \mathrm{i}y$, the standard notation for complex numbers. Numbers of the form $\mathrm{i}y$, $y \in \mathbb{R}$, are called (purely) *imaginary* numbers. Note that $\mathrm{i}^2 = -1$.

Complex numbers are not ordered. Still we will encounter inequalities like $z \geq w$ and, if we do, it is tacitly assumed that $z, w \in \mathbb{R}$.

1.1.3 Real and imaginary parts of a complex number. If $a, b \in \mathbb{R}$ and $z = a + \mathrm{i}b$ is a complex number, then a is called the *real part* of z (denoted by $\operatorname{Re} z$) and b is called the *imaginary part* of z (denoted by $\operatorname{Im} z$). The number $\overline{z} = a - \mathrm{i}b$ is called the *complex conjugate* of z. If $z \in \mathbb{C}$, then $z + \overline{z} = 2 \operatorname{Re} z$, $z - \overline{z} = 2\mathrm{i} \operatorname{Im} z$, and $z\overline{z} = (\operatorname{Re} z)^2 + (\operatorname{Im} z)^2 \geq 0$. If $z, w \in \mathbb{C}$, then $\overline{z + w} = \overline{z} + \overline{w}$ and $\overline{zw} = \overline{z}\,\overline{w}$.

1.1.4 Absolute value of a complex number. If $a, b \in \mathbb{R}$ and $z = a + \mathrm{i}b$ is a complex number, then $|z| = \sqrt{a^2 + b^2} \in [0, \infty)$ is called the *absolute value* or *modulus* of z. Note that this definition is compatible with the definition of the absolute value of a real number.

If $z \in \mathbb{C}$, then $|z|^2 = z\overline{z}$, $|\operatorname{Re} z| \le |z|$, and $|\operatorname{Im} z| \le |z|$. If $z \ne 0$, then $1/z = \overline{z}/|z|^2$. The absolute value of a product of two complex numbers equals the product of the absolute values of the numbers, i.e., $|z_1 z_2| = |z_1|\,|z_2|$.

1.1.5 The triangle inequality. If $x, y \in \mathbb{C}$, then

$$|x + y| \le |x| + |y| \quad \text{and} \quad |x + y| \ge \big||x| - |y|\big|.$$

1.2. The topology and geometry of complex numbers

1.2.1 Distance between complex numbers. Given two complex number z_1 and z_2 we call $|z_1 - z_2|$ their *distance*. In fact, equipped with this distance function, $\mathbb{C}$ becomes a metric space and thus an object of geometry. When one wants to emphasize the geometry of $\mathbb{C}$ one calls it the *complex plane*. Appendix A gathers some pertinent information about metric (and topological) spaces. In particular, it reviews the concepts of convergence and limit in metric spaces, which are obvious generalizations of those in $\mathbb{R}$.

1.2.2 Polar representation of complex numbers. For any non-zero complex number $z = x + \mathrm{i}y$, $x, y \in \mathbb{R}$ define $r = \sqrt{x^2 + y^2} = |z| > 0$ and $\theta \in (-\pi, \pi]$ by the relations $\cos\theta = x/r$ and $\sin\theta = y/r$. These requirements determine r and θ uniquely. Interpreting z as a point (x, y) in a two-dimensional rectangular coordinate system the number r represents the distance of z from the origin. If $\theta \ge 0$, it is the counterclockwise angle from the positive x-axis to the ray connecting z and the origin, otherwise $-\theta$ is the clockwise angle. The numbers r and θ are called the *polar coordinates* of z.

1.2.3 Powers and roots. For any $z \in \mathbb{C}$ we define $z^0 = 1$ and, inductively, $z^n = zz^{n-1}$ for $n \in \mathbb{N}$. When $z \ne 0$ and $n \in \mathbb{N}$ we note that $(z^{-1})^n = (z^n)^{-1}$ and we abbreviate this number by z^{-n}. The numbers z^n, $n \in \mathbb{Z}$, are called the (integer) *powers* of z.

We also define *n-th roots* of complex numbers when $n \in \mathbb{N}$. The complex number a is called an n-th root of a complex number b, if $a^n = b$. Every non-zero complex number has exactly n n-th roots. To see this note that $(\cos\theta + \mathrm{i}\sin\theta)^n = \cos(n\theta) + \mathrm{i}\sin(n\theta)$ when $n \in \mathbb{N}$.

1.2.4 Open and closed sets. As a metric space $\mathbb{C}$ has open and closed balls which, in this context, are usually called disks. Thus, as explained in A.1.5, $\mathbb{C}$ is a topological space whose open sets are unions of open disks; in particular open disks are open. The following characterizations of open and closed sets are often very useful.

- A set $S \subset \mathbb{C}$ is open if and only if for every $x \in S$ there is an $r > 0$ such that $B(x, r) \subset S$.
- A set $S \subset \mathbb{C}$ is closed if and only if for every convergent sequence whose elements lie in S the limit also lies in S.

From this one may conclude that closed disks are closed.

As a metric space $\mathbb{C}$ is the same as $\mathbb{R}^2$. Hence, by the Heine-Borel theorem A.2.5, a subset of the complex plane is compact precisely when it is closed and bounded.

1.2.5 Line segments and convex sets. Given two points z and w in $\mathbb{C}$ we call the set $\{(1-t)z + tw : 0 \le t \le 1\}$ the *line segment* joining z and w. The number $|z - w|$ is called the *length* of the segment.

A subset S of $\mathbb{C}$ is called *convex* if it contains the line segment joining x and y whenever $x, y \in S$. Every disk is convex.

1.2.6 Connectedness. A subset S of $\mathbb{C}$ is called *connected* if $S \cap A \cap B \neq \emptyset$ whenever $A, B \subset \mathbb{C}$ are open sets neither of which covers S but whose union does.

The empty set, all singletons, and all convex sets are connected. To prove the last claim assume that the convex set S is not connected and pick points $x \in S \cap A$ and $y \in S \cap B$. The fact that $\{t \in [0,1] : (1-t)x + ty \in A\}$ has a supremum leads to a contradiction.

1.2.7 Connected components. If S is a subset of the complex plane and $x \in S$, let $\mathcal{C}(x)$ be the collection of all connected subsets of S which contain x. Any union of sets in $\mathcal{C}(x)$ is connected. In particular, $T = \bigcup_{C \in \mathcal{C}(x)} C$ is a connected subset of S. Any larger subset of S is not connected. T is called a component of S. The components of S are pairwise disjoint and their union equals S.

The components of open sets are open. There are at most countably many of them.

1.3. Sequences and series

1.3.1 Numerical sequences. A sequence $n \mapsto z_n$ of complex numbers converges[1] if and only if both of the real sequences $n \mapsto \mathrm{Re}(z_n)$ and $n \mapsto \mathrm{Im}(z_n)$ converge. In this case

$$\lim_{n\to\infty} z_n = \lim_{n\to\infty} \mathrm{Re}(z_n) + \mathrm{i} \lim_{n\to\infty} \mathrm{Im}(z_n).$$

This result allows to extend immediately many theorems proved for sequences of real numbers to sequences of complex numbers. In particular, if the sequences z and w have limits, then so do $z \pm w$ and zw. In fact,

$$\lim_{n\to\infty} (z_n \pm w_n) = \lim_{n\to\infty} z_n \pm \lim_{n\to\infty} w_n,$$
$$\lim_{n\to\infty} z_n w_n = \lim_{n\to\infty} z_n \lim_{n\to\infty} w_n,$$

and, assuming also $w_n \neq 0$ for all n and $\lim_{n\to\infty} w_n \neq 0$,

$$\lim_{n\to\infty} \frac{z_n}{w_n} = \frac{\lim_{n\to\infty} z_n}{\lim_{n\to\infty} w_n}.$$

1.3.2 Cauchy sequences. A sequence z of complex numbers is called a *Cauchy*[2] *sequence*, if

$$\forall \varepsilon > 0 : \exists N > 0 : \forall n, m > N : |z_n - z_m| < \varepsilon.$$

THEOREM. A sequence of complex numbers converges if and only if it is a Cauchy sequence.

1.3.3 Series. If $z : \mathbb{N} \to \mathbb{C}$ is a sequence of complex numbers, the sequence

$$s : \mathbb{N} \to \mathbb{C} : n \mapsto s_n = \sum_{k=1}^{n} z_k$$

is a called the *sequence of partial sums* of z or a *series*. We will denote s by $\sum_{k=1}^{\infty} z_k$.

If the sequence s of partial sums of z converges to $L \in \mathbb{C}$, we say the series $\sum_{k=1}^{\infty} z_k$ *converges* to L. We then write (abusing notation slightly)

$$\sum_{k=1}^{\infty} z_k = \lim_{n\to\infty} s_n = L.$$

[1]Refer to A.1.6 for the definition of convergence.

[2]Augustin-Louis Cauchy (1789 – 1857)

Of course, a series (i.e., a sequence of partial sums) may also *diverge*.

1.3.4 Absolute convergence of series. The series $\sum_{k=1}^{\infty} z_k$ is called *absolutely convergent* if $\sum_{k=1}^{\infty} |z_k|$ is convergent. Every absolutely convergent series is convergent.

Suppose that $\sum_{n=1}^{\infty} z_n$ converges absolutely and let $\sum_{n=1}^{\infty} w_n$ be a rearrangement of $\sum_{n=1}^{\infty} z_n$. Then $\sum_{n=1}^{\infty} w_n$ converges absolutely, and

$$\sum_{n=1}^{\infty} w_n = \sum_{n=1}^{\infty} z_n.$$

This follows from 1.3.1 since the analogous statement is known to hold for series of real numbers.

1.3.5 The geometric series. If $|z| < 1$, then the *geometric series* $\sum_{n=0}^{\infty} z^n$ converges absolutely. Its limit is $1/(1-z)$.

CHAPTER 2

Complex-valued functions of a complex variable

2.1. Limits and continuity

2.1.1 Limits. Let S be a subset of $\mathbb{C}$, $f : S \to \mathbb{C}$ a function, and $a \in \mathbb{C}$ a limit point of S. We say that f *converges* to the complex number b as z tends to a if the following statement is true:

$$\forall \varepsilon > 0 : \exists \delta > 0 : \forall z \in S : 0 < |z - a| < \delta \Rightarrow |f(z) - b| < \varepsilon.$$

Note that this definition can easily be extended to any metric space.

As for sequences, f can converge to at most one value as z tends to a. This value is then called the *limit* of f as z tends to a. It is denoted by $\lim_{z \to a} f(z)$.

The results on limits of sequences stated in 1.3.1 hold analogously also for functions.

2.1.2 Continuity. Let S be a subset of $\mathbb{C}$ and $a \in S$. A function $f : S \to \mathbb{C}$ is continuous at a if the following statement is true:

$$\forall \varepsilon > 0 : \exists \delta > 0 : \forall z \in S : |z - a| < \delta \Rightarrow |f(z) - f(a)| < \varepsilon.$$

Any function is continuous at any isolated point of its domain. Suppose $S \subset \mathbb{C}$, $f : S \to \mathbb{C}$, and $a \in S$ is a limit point of S. Then f is continuous at a if and only if $\lim_{z \to a} f(z)$ exists and equals $f(a)$. It follows that sums and products and compositions of continuous functions are continuous.

Important examples of continuous functions (on all of $\mathbb{C}$) are Re, Im and $|\cdot|$.

2.2. Holomorphic functions

2.2.1 Differentiation. Let S be a subset of $\mathbb{C}$, $f : S \to \mathbb{C}$ a function, and $a \in S$ a limit point of S. We say that f is *differentiable* at a if

$$\lim_{z \to a} \frac{f(z) - f(a)}{z - a}$$

exists. The limit is called the *derivative* of f at a and is commonly denoted by $f'(a)$. If $f : S \to \mathbb{C}$ is differentiable at every point $a \in S$, we say that f is differentiable on S.

This definition is completely analogous to that of a derivative of a function of a real variable. In fact, when S is a real interval (one of the cases we are interested in) things are hardly different from Real Analysis; the fact that f assumes complex values is not very important. However, if S is an open set and z may approach a from many directions, the existence of the limit has far-reaching consequences.

2.2.2 Basic properties of derivatives. Suppose S, S_1 and S_2 are subsets of $\mathbb{C}$. Then the following statements hold true:

(1) A function $f : S \to \mathbb{C}$ is differentiable at a point $a \in S$ if and only if there is a number F and a function $h : S \to \mathbb{C}$ which is continuous at a, vanishes there, and satisfies $f(z) = f(a) + F(z - a) + h(z)(z - a)$. In this case we have $F = f'(a)$.

(2) If $f : S \to \mathbb{C}$ is differentiable at $a \in S$, then it is continuous at a.
(3) Sums and products of differentiable functions with common domains are again differentiable.
(4) The chain rule holds, i.e., if $f : S_1 \to S_2$ is differentiable at a and $g : S_2 \to \mathbb{C}$ is differentiable at $f(a)$, then $g \circ f$ is differentiable at a, and $(g \circ f)'(a) = g'(f(a))f'(a)$.

2.2.3 Differentiation with respect to real and complex variables. Suppose S is a real interval. Then the function $f : S \to \mathbb{C}$ is differentiable at $a \in S$ if and only if $\operatorname{Re} f$ and $\operatorname{Im} f$ are differentiable there. In this case, $f'(a) = (\operatorname{Re} f)'(a) + \mathrm{i}(\operatorname{Im} f)'(a)$.

The situation is completely different if S is an open set in $\mathbb{C}$. For instance, the function $z \mapsto f(z) = z^2$ is differentiable on all of $\mathbb{C}$ and $f'(z) = 2z$. However, its real and imaginary parts are differentiable only at $z = 0$. In fact, if a real-valued function defined in $B(a, r)$ is differentiable at a, then its derivative must be zero.

2.2.4 Holomorphic functions. Let Ω be a non-empty open set. A function $f : \Omega \to \mathbb{C}$ is called *holomorphic* on Ω if it is differentiable at every point of Ω. A function which is defined and holomorphic on all of $\mathbb{C}$ is called *entire*.

It is easy to think that the notions of differentiability and holomorphicity are the same. To avoid this mistake note that differentiability is a pointwise concept while holomorphicity is defined on open sets. Nevertheless, we might say that a function is holomorphic at a point, if it is holomorphic on a neighborhood of that point.

2.2.5 Basic properties of holomorphic functions. Holomorphic functions are continuous on their domain.

Sums, differences, and products of holomorphic functions (on common domains) are holomorphic. The composition of holomorphic functions is also holomorphic. The usual formulas hold, including the chain rule.

All polynomials (of a single variable) are entire functions. The function $z \mapsto 1/z$ is holomorphic on $\mathbb{C} \setminus \{0\}$. The quotient f/g of two holomorphic functions $f, g : \Omega \to \mathbb{C}$ is holomorphic on $\Omega \setminus \{z \in \Omega : g(z) = 0\}$.

2.3. Integration

2.3.1 Integrals of complex-valued functions over intervals. Let $[a, b]$ be a closed interval in $\mathbb{R}$ and f a complex-valued function on $[a, b]$. Then $\operatorname{Re} f$ and $\operatorname{Im} f$ are real-valued functions on $[a, b]$. We say that f is *Riemann integrable* over $[a, b]$ if $\operatorname{Re} f$ and $\operatorname{Im} f$ are. The integral is defined to be

$$\int_a^b f = \int_a^b \operatorname{Re} f + \mathrm{i} \int_a^b \operatorname{Im} f.$$

The integral is linear and satisfies

$$\left| \int_a^b f \right| \leq \int_a^b |f|.$$

To prove this inequality let $z = \int_a^b f$ and, when this is different from 0, set $\alpha = |z|/z$. Then consider $\int_a^b \alpha f$ which is real.

2.3.2 Paths. Let $\Omega \subset \mathbb{C}$ be a non-empty open set and $[a, b]$ a non-trivial bounded interval in $\mathbb{R}$. A continuous function $\gamma : [a, b] \to \Omega$ is then called a *path* in Ω. If the derivative γ' exists and is continuous on $[a, b]$, γ is called a *smooth path*.

The points $\gamma(a)$ and $\gamma(b)$ are called *initial point* and *end point* of γ, respectively. A path is called closed if its initial and end points coincide. The range of γ, i.e., the set $\{\gamma(t) : t \in [a,b]\}$ is denoted by γ^*. Note that γ^* is compact and connected. The number $\int_a^b |\gamma'(t)|dt$ is called the *length* of the smooth path γ.

Let $\gamma : [a,b] \to \Omega$ be a smooth path. The coordinate transform $\varphi(t) = (1-t)a + tb$, $t \in [0,1]$, gives rise to a new smooth path $\tilde{\gamma} = \gamma \circ \varphi$ defined on $[0,1]$ with the same initial and end points and the same range as γ. Similarly, $\eta(t) = \gamma(a+b-t)$, $t \in [a,b]$, called the *opposite* of γ, also has the same range as γ but switches initial and end points. We use the notation $\eta = \ominus\gamma$.

2.3.3 Integrals along smooth paths. Suppose $\gamma : [a,b] \to \mathbb{C}$ is a smooth path in $\mathbb{C}$ and $f : \gamma^* \to \mathbb{C}$ is continuous. Then the number

$$\int_a^b (f \circ \gamma)\gamma'$$

is well-defined. It is called the integral of f along γ and denoted by $\int_\gamma f$. We have the estimate

$$\left| \int_\gamma f \right| \le \sup(|f|(\gamma^*))L(\gamma)$$

where $L(\gamma)$ is the length of γ.

Suppose f is a continuous function defined on γ^* and $\tilde{\gamma}(t) = \gamma((1-t)a + tb)$. Then $\int_{\ominus\gamma} f = -\int_\gamma f$ and $\int_{\tilde{\gamma}} f = \int_\gamma f$. The latter identity shows that, when we are interested in integrals, we may always choose a parametrization with any fixed interval as domain.

2.3.4 Example. Let $\gamma : [0, 2\pi] \to \mathbb{C} : t \mapsto \cos(t) + \mathrm{i}\sin(t)$ and $f : \mathbb{C} \setminus \{0\} \to \mathbb{C} : z \mapsto 1/z$. Then $\int_\gamma f = 2\pi\mathrm{i}$.

2.4. Contours

2.4.1 Contours. A *contour* in Ω is a finite ordered list of smooth paths in the non-empty open set Ω. Among the contours we introduce an associative binary relation $\oplus$ which assigns to a and b, lists of length m and n, respectively, the concatenation of these lists, denoted by $a \oplus b$, i.e., a list of length $m+n$. We will write $\gamma_1 \ominus \gamma_2$ for $\gamma_1 \oplus (\ominus\gamma_2)$.

Suppose $\Gamma = \bigoplus_{k=1}^n \gamma_k$ is a contour in Ω. Then we denote the compact set $\bigcup_{k=1}^n \gamma_k^*$ by Γ^*. Moreover, if f is continuous on Γ^* we define

$$\int_\Gamma f = \sum_{k=1}^n \int_{\gamma_k} f.$$

2.4.2 Closed contours. A contour $\gamma_1 \oplus ... \oplus \gamma_n$ is called *closed* when there is a permutation π of $\{1,...,n\}$ such that the end point of γ_k coincides with the initial point of $\gamma_{\pi(k)}$ for $k = 1,...,n$.

2.4.3 Connected contours. A particularly important instance of a contour $\Gamma = \gamma_1 \oplus ... \oplus \gamma_n$ is when for $k = 1,...,n-1$, the end point of γ_k coincides with the initial point of γ_{k+1} (perhaps after reordering the indices). A contour of this type is called a *connected contour*. The initial point of γ_1 is called the *initial point* of Γ while the end point of γ_n is called the *end point* of Γ. We say a contour connects x to y, if it is a connected contour with initial point x and end point y.

If Γ is a connected contour, then Γ^* is connected.

THEOREM. Let S be a subset of $\mathbb{C}$. If any two points of S are connected by a connected contour in S, then S is connected. Conversely, if S is open and connected and x, y are two points in S, then there is a connected contour in S which connects x and y.

SKETCH OF PROOF. The first claim follows from 1.2.7. For the second claim let $C(x)$ denote the set of all $y \in S$ for which there is a contour in S connecting some fixed point x and y. If S is open, then so are $C(x)$ and $S \setminus C(x)$. □

2.4.4 Polygonal contours. Let $z_1, ..., z_{n+1} \in \mathbb{C}$. We define $\gamma_k(t) = (1-t)z_k + tz_{k+1}$ for $t \in [0,1]$ and $k = 1, ..., n$. Then γ_k is a smooth path with initial point z_k and end point z_{k+1}. Note that γ_k^* is the line segment joining z_k and z_{k+1}. The length of the line segment equals the length of γ_k. Moreover, $\gamma_1 \oplus ... \oplus \gamma_n$ is a connected contour which we denote by $\langle z_1, ..., z_{n+1} \rangle$. For instance, when $n = 3$ and $z_4 = z_1$, $\Gamma = \langle z_1, z_2, z_3, z_1 \rangle$ is a closed contour tracing the circumference of a (possibly degenerate) *triangle*.

2.4.5 Primitives. Suppose Ω is a non-empty open set and $F' = f$ on Ω. Then F is called a *primitive* of f. If F is a primitive of f and c is a constant, then $F + c$ is also a primitive of f.

If $\gamma : [a,b] \to \Omega$ is a smooth path and the continuous function f has a primitive F in Ω, then $\int_\gamma f = F(\gamma(b)) - F(\gamma(a))$.

The primitives of the zero function in a non-empty connected open set are precisely the constant functions.

2.5. Series of functions

2.5.1 Pointwise and uniform convergence. Let $S \subset \mathbb{C}$ and, for each $n \in \mathbb{N}$, let f_n be a function from S to $\mathbb{C}$. The map $n \mapsto f_n$ is called a *sequence* of functions. We say that $n \mapsto f_n$ *converges pointwise* to a function $f : S \to \mathbb{C}$ if for *each* point $z \in S$ the numerical sequence $n \mapsto f_n(z)$ converges to $f(z)$. We say that $n \mapsto f_n$ *converges uniformly* to a function $f : S \to \mathbb{C}$ if, loosely speaking, the rate of convergence does not depend on $z \in S$. More precisely, $n \mapsto f_n$ converges to f pointwise, if

$$\forall z \in S : \forall \varepsilon > 0 : \exists N \in \mathbb{R} : \forall n > N : |f_n(z) - f(z)| < \varepsilon$$

is true. On the other hand, $n \mapsto f_n$ converges to f uniformly, if

$$\forall \varepsilon > 0 : \exists N \in \mathbb{R} : \forall z \in S : \forall n > N : |f_n(z) - f(z)| < \varepsilon$$

holds. Note the order of the quantifiers in these statements.

Obviously, uniform convergence implies pointwise convergence but not vice versa.

A series of functions is defined as the sequence of partial sums of functions with a common domain. Thus the definitions of pointwise and uniform convergence extend also to series of functions.

2.5.2 Cauchy criterion for uniform convergence. Let $S \subset \mathbb{C}$ and $f_n : S \to \mathbb{C}$ for $n \in \mathbb{N}$. The sequence of functions $n \mapsto f_n$ converges uniformly on S if and only if for all $\varepsilon > 0$ there exists $N \in \mathbb{R}$ such that for all $n, m > N$ and all $z \in S$ it holds that $|f_n(z) - f_m(z)| < \varepsilon$.

2.5.3 The Weierstrass[1] M-test. Let $S \subset \mathbb{C}$ and suppose $n \mapsto g_n$ is a sequence of complex-valued functions defined on S. Assume that there are non-negative numbers M_n

[1] Karl Weierstraß (1815 – 1897)

such that $|g_n(z)| \leq M_n$ for all $z \in S$ and that the series $\sum_{n=1}^{\infty} M_n$ converges. Then the following statements hold:

(1) The series $\sum_{n=1}^{\infty} g_n(z)$ converges absolutely for every $z \in S$.
(2) The series $\sum_{n=1}^{\infty} g_n$ converges uniformly in S.

2.5.4 Uniform convergence and continuity. Let $S \subset \mathbb{C}$ and $n \mapsto f_n$ a sequence of continuous complex-valued functions on S which converges uniformly to a function $f : S \to \mathbb{C}$. Then f is continuous on S.

2.5.5 Uniform convergence and integration. Let $[a, b]$ be a bounded interval in $\mathbb{R}$. Suppose that $n \mapsto f_n : [a, b] \to \mathbb{C}$ is a sequence of functions which are Riemann integrable over $[a, b]$ and that this sequence converges uniformly to a function $f : [a, b] \to \mathbb{C}$. Then f is Riemann integrable and

$$\int_a^b f = \lim_{n\to\infty} \int_a^b f_n.$$

Of course, the analogous result holds for series: Suppose that $n \mapsto g_n : [a, b] \to \mathbb{C}$ is a sequence of functions which are Riemann integrable over $[a, b]$ and that the series $\sum_{n=0}^{\infty} g_n$ converges uniformly to a function $g : [a, b] \to \mathbb{C}$. Then g is Riemann integrable and

$$\int_a^b g = \sum_{n=0}^{\infty} \int_a^b g_n.$$

2.6. Analytic functions

2.6.1 Power Series. Let a_n, $n \in \mathbb{N}_0$, be a sequence of complex numbers. Then the series $\sum_{n=0}^{\infty} a_n z^n$, $z \in \mathbb{C}$, is called a *power series*. If the series converges for every $z \in \mathbb{C}$ we say that it has infinite radius of convergence. Otherwise there is a non-negative number R such that the series converges absolutely whenever $|z| < R$ and diverges whenever $|z| > R$. The number R is called the *radius of convergence* of the series. The disk $B(0, R)$ is called the disk of convergence.

If k is a non-negative integer than the series $\sum_{n=0}^{\infty} n^k a_n z^n$ have the same radius of convergence regardless of k.

2.6.2 Power series define holomorphic functions. A power series with a positive or infinite radius of convergence defines a holomorphic function in its disk of convergence. In fact, if $f(z) = \sum_{n=0}^{\infty} a_n(z - z_0)^n$ for $|z - z_0| < R$, then $f'(z) = \sum_{n=1}^{\infty} na_n(z - z_0)^{n-1}$ in the same disk, i.e., the power series may be differentiated term by term and f is holomorphic on $B(z_0, R)$.

Sketch of proof. Let $g(w) = \sum_{n=1}^{\infty} na_n(w-z_0)^{n-1}$ which converges for $w \in B(z_0, R)$ by 2.6.1. Set $r = \max\{|z - z_0|, |w - z_0|\}$ and use twice the identity

$$a^n - b^n = (a - b) \sum_{k=0}^{n-1} a^k b^{n-k-1}$$

with $a = z - z_0$ and $b = w - z_0$ to show that

$$\left| \frac{f(z) - f(w)}{z - w} - g(w) \right| \leq |z - w| \sum_{n=2}^{\infty} \frac{n(n - 1)}{2} |a_n| r^{n-2}.$$

□

2.6.3 Analytic functions. We say that a function $f : B(z_0, R) \to \mathbb{C}$ is represented by a power series if $f(z) = \sum_{n=0}^{\infty} a_n(z - z_0)^n$ for all $z \in B(z_0, R)$.

Let Ω be a non-empty open set. A function $f : \Omega \to \mathbb{C}$ is called *analytic* in Ω, if Ω is a union of open disks in each of which f is represented by a power series.

For example, $z \mapsto 1/(1-z)$ is analytic in the (open) unit disk as well as in all of $\mathbb{C} \setminus \{1\}$.

THEOREM. Every analytic function is holomorphic.

The central and most astonishing theorem of complex analysis is that the converse is also true, i.e., every holomorphic function is analytic, see Theorem 3.3.2 below.

2.6.4 Taylor series. Suppose $f(z) = \sum_{n=0}^{\infty} a_n(z-z_0)^n$ for $|z-z_0| < R$, i.e., $f : B(z_0, R) \to \mathbb{C}$ is analytic. Then f is infinitely often differentiable and $a_n = f^{(n)}(z_0)/n!$. We call

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!}(z - z_0)^n$$

the Taylor[2] series of f about z_0.

2.6.5 The exponential function. The series

$$\exp(z) = \sum_{n=0}^{\infty} \frac{z^n}{n!}$$

converges absolutely for every $z \in \mathbb{C}$ and uniformly in every closed disk about the origin. It represents therefore an entire function, called the *exponential function*. It is an extension of the exponential function defined in Real Analysis.

THEOREM. If $t \in \mathbb{R}$, then $\exp(\mathrm{i}t) = \cos(t) + \mathrm{i}\sin(t)$; in particular, $\exp(0) = 1$. Moreover $\exp'(z) = \exp(z)$, and $\exp(a + b) = \exp(a)\exp(b)$. The exponential function has no zeros. It is periodic with period $2\pi\mathrm{i}$. Any period of the exponential function is an integer multiple of $2\pi\mathrm{i}$. In particular, $\exp(z) = 1$ if and only if z is an integer multiple of $2\pi\mathrm{i}$.

2.6.6 Analytic functions defined by integrals. Suppose ψ and ϕ are continuous complex-valued functions on $[a, b]$. Then the function f defined by

$$f(z) = \int_a^b \frac{\psi(t)}{\phi(t) - z} dt$$

is analytic in $\Omega = \mathbb{C} \setminus \phi([a, b])$. In fact, if $z_0 \in \Omega$, if $r = \inf\{|\phi(t) - z_0| : t \in [a, b]\}$, and if $z \in B(z_0, r)$, then

$$f(z) = \sum_{n=0}^{\infty} a_n(z - z_0)^n$$

where

$$a_n = \int_a^b \frac{\psi(t)dt}{(\phi(t) - z_0)^{n+1}}.$$

SKETCH OF PROOF. Since $\phi(t) - z = (\phi(t) - z_0)(1 - (z - z_0)/(\phi(t) - z_0))$ we may use properties of the geometric series which is uniformly convergent on appropriate disks. □

[2]Brook Taylor (1685 – 1731)

CHAPTER 3

Cauchy's theorem and some of its consequences

3.1. The index of a point with respect to a closed contour

3.1.1. Let Γ be a contour in $\mathbb{C}$ and $\Omega = \mathbb{C} \setminus \Gamma^*$. Since Γ^* is compact Ω is open. Ω has precisely one unbounded connected component. This component contains the set $\overline{B}(0, R)^c$ when R is chosen so that $\Gamma^* \subset \overline{B}(0, R)$.

3.1.2. If $\gamma : [a, b] \to \mathbb{C}$ is a smooth path and $z \in \mathbb{C} \setminus \gamma^*$, define $F : [a, b] \to \mathbb{C}$ by

$$F(s) = \exp\left(\int_a^s \frac{\gamma'(t)}{\gamma(t) - z} dt\right).$$

Then $s \mapsto F(s)/(\gamma(s) - z)$ is constant.

3.1.3 The index of a point with respect to a closed contour. Let Γ be a closed contour in $\mathbb{C}$ and $\Omega = \mathbb{C} \setminus \Gamma^*$. The function $\operatorname{Ind}_\Gamma$ defined by

$$\operatorname{Ind}_\Gamma(z) = \frac{1}{2\pi \mathrm{i}} \int_\Gamma \frac{du}{u - z}$$

on Ω is an analytic function assuming only integer values. It is constant on each connected component of Ω and, in particular, zero near infinity.

Sketch of proof. Let $\Gamma = \gamma_1 \oplus ... \oplus \gamma_n$ where the γ_k are smooth paths defined on $[a_k, b_k]$, respectively. Using 2.6.6 we see that $\operatorname{Ind}_\Gamma$ is analytic. Now define F_k for each γ_k as in 3.1.2 so that $F_k/(\gamma_k - z)$ is constant. In particular, $F_k(b_k) = (\gamma_k(b_k) - z)/(\gamma_k(a_k) - z)$. Hence

$$\exp(2\pi \mathrm{i} \operatorname{Ind}_\Gamma(z)) = \prod_{k=1}^n \frac{\gamma_k(b_k) - z}{\gamma_k(a_k) - z} = 1.$$

By Theorem 2.6.5 $\operatorname{Ind}_\Gamma(z)$ is an integer. Continuity shows that the value of $\operatorname{Ind}_\Gamma$ is constant on sufficiently small disks within Ω. Hence the sets $\{z \in \Omega : \operatorname{Ind}_\Gamma(z) = k\}$ are open for every $k \in \mathbb{Z}$. □

3.1.4 Example. Consider the smooth paths $\gamma_1(t) = \exp(\mathrm{i}t)$, $\gamma_2(t) = \exp(\mathrm{i}t)/2$, and $\gamma_3(t) = \exp(-\mathrm{i}t)/2$ all defined on $[0, 2\pi]$ and the contours $\Gamma_1 = \gamma_1$, $\Gamma_2 = \gamma_1 \oplus \gamma_2$, and $\Gamma_3 = \gamma_1 \oplus \gamma_3$. The contours are closed so that $\operatorname{Ind}_{\Gamma_k}(z)$ is defined for $z = 0$, $z = 3\mathrm{i}/4$, and $z = -2$.

The number $\operatorname{Ind}_\Gamma(z)$ is also called the *winding number* of the contour Γ around z. The above examples hint at the origin of the name.

3.2. Cauchy's theorem

3.2.1 Cauchy's theorem for functions with primitives. Suppose Ω is a non-empty open set, f a continuous function on Ω which has a primitive F. If Γ is a closed contour in Ω, then $\int_\Gamma f = 0$.

Sketch of proof. Suppose $\Gamma = \gamma_1 \oplus ... \oplus \gamma_n$ with smooth paths γ_k defined on $[a_k, b_k]$. Evaluate $\int_{\gamma_k} f$ using 2.4.5 and recall that $\gamma_k(b_k) = \gamma_{\pi(k)}(a_{\pi(k)})$ for some permutation π of the set $\{1, ..., n\}$. □

3.2.2 Cauchy's theorem for integer powers. Suppose $n \in \mathbb{Z}$. Unless $n = -1$ the power function $z \mapsto z^n$ has the primitive $z \mapsto z^{n+1}/(n+1)$ in either $\Omega = \mathbb{C} \setminus \{0\}$ or $\Omega = \mathbb{C}$ depending on whether n is negative or not. Consequently, if Γ is a closed contour in Ω and if $n \in \mathbb{Z} \setminus \{-1\}$, then $\int_\Gamma z^n dz = 0$.

The exceptional case $n = -1$ gives rise to interesting complications which we will discuss later. Recall, though, from 2.3.4 that $\int_\gamma dz/z = 2\pi \mathrm{i}$ when $\gamma(t) = \exp(\mathrm{i}t)$ for $t \in [0, 2\pi]$.

3.2.3 Triangles. Let a, b, c be three pairwise distinct complex numbers. The set $\Delta = \{t_1 a + t_2 b + t_3 c : t_j \in [0,1], t_1 + t_2 + t_3 = 1\}$ is called a solid *triangle* with *vertices* a, b, and c. The line segments joining each pair of vertices are called *edges* of the triangle. They are obtained by setting t_1, t_2, and t_3 in turn equal to zero and are thus subsets of Δ.

The diameter of a triangle is the length of its longest edge. To show this set $D = \max\{|c-b|, |b-a|, |a-c|\}$, $x = t_1 a + t_2 b + t_3 c$, and $y = s_1 a + s_2 b + s_3 c$. Note that $|x - y| = |(t_1 - s_1)(a - c) + (t_2 - s_2)(b - c)| = |(t_2 - s_2)(b - a) + (t_3 - s_3)(c - a)| = |(t_3 - s_3)(c - b) + (t_1 - s_1)(a - b)|$. Two of the numbers $t_1 - s_1$, $t_2 - s_2$, and $t_3 - s_3$ are of the same sign unless one of them is zero.

3.2.4 Quartering a triangle. Let Δ be a solid triangle with vertices a, b, and c. If $c' = (a+b)/2$, $b' = (a+c)/2$, and $a' = (b+c)/2$ (the midpoints of the edges), the sets $\{a', b', c'\}$, $\{a, b', c'\}$, $\{a', b, c'\}$, and $\{a', b', c\}$ define respectively four new triangles Δ_0, ..., Δ_3. Their union is Δ and the intersection of any two of these is either a midpoint a', b', or c' or one of the segments joining two midpoints. To show this note that $\Delta_0 = \{t_1 a + t_2 b + t_3 c : t_j \in [0, 1/2], t_1 + t_2 + t_3 = 1\}$ and $\Delta_1 = \{t_1 a + t_2 b + t_3 c : t_1 \in [1/2, 1], t_2, t_3 \in [0, 1/2], t_1 + t_2 + t_3 = 1\}$ with similar expressions for Δ_2 and Δ_3.

Also Δ_0, ..., Δ_3 are congruent and are similar to Δ itself. The circumference and the diameter of each one of the Δ_k is half of that of Δ.

3.2.5 Integration along a triangle. Suppose a, b, c are pairwise distinct complex numbers. Suppose $a' \in \langle b, c\rangle^*$, $b' \in \langle c, a\rangle^*$, and $c' \in \langle a, b\rangle^*$. Let $\Gamma = \langle a, b, c, a\rangle$ and $\Lambda = \langle a, c', b', a\rangle \oplus \langle b, a', c', b\rangle \oplus \langle c, b', a', c\rangle \oplus \langle c', a', b', c'\rangle$. If f is continuous on Λ^*, then $\int_\Gamma f = \int_\Lambda f$.

3.2.6 Solid triangles are compact. To prove this claim let us first consider the solid triangle with vertices 1, i and 0. The points of the sequence $z_n = s_n + \mathrm{i} t_n$ are in the triangle precisely when s_n, t_n, and $u_n = 1 - s_n - t_n$ are in $[0, 1]$. Assuming that z_n is convergent, 1.3.1 gives that s_n and t_n and hence u_n are convergent. Their limits s, t, and u are in $[0, 1]$ and satisfy $s + t + u = 1$ implying that z_n converges to a point in the triangle. Thus the triangle is a closed set. It is, of course, bounded and, by the Heine-Borel theorem, compact.

A general triangle is the image of this triangle under a continuous map and also compact according to A.2.2.

3.2.7 Cauchy's theorem for triangles. Suppose Ω is a non-empty open set, z_0 a point in Ω, and $f : \Omega \to \mathbb{C}$ a continuous function which is holomorphic on $\Omega \setminus \{z_0\}$. If Δ is a solid triangle in Ω with vertices a, b, and c, let $\Gamma = \langle a, b, c, a\rangle$. Then $\int_\Gamma f = 0$.

Sketch of proof. Assume first that $z_0 \notin \Delta$. Let $\Delta_0 = \Delta$. Given Δ_n let Δ_{n+1} be one of the four triangles constructed in 3.2.3. Then $\bigcap_{n=1}^\infty \Delta_n$ consists of one point only, say w.

By 2.2.2 the function h defined by $f(z) = f(w) + f'(w)(z-w) + h(z)(z-w)$ is continuous and vanishes at w. Hence, given $\varepsilon > 0$ and using 3.2.2 and 2.3.3, there is an n such that $|\int_{\Gamma_n} f| \leq 3\varepsilon D^2 4^{-n}$ when D denotes the diameter of Δ. By making, at each step, the right choice of triangle in the above recursion one can also show that $|\int_{\Gamma_n} f| \geq 4^{-n}|\int_\Gamma f|$. This settles the first case. If z_0 is a vertex of Δ, we repeat the construction above but choosing Δ_n always to be the one which has z_0 as a vertex so that $\int_\Gamma f = \int_{\Gamma_n} f$. Finally, if $z_0 \in \Delta \setminus \{a, b, c\}$ construct three triangles in Δ which have z_0 as a vertex. □

3.2.8 Cauchy's theorem for convex sets. Suppose Ω is a non-empty open convex set, z_0 a point in Ω, and $f : \Omega \to \mathbb{C}$ a continuous function which is holomorphic on $\Omega \setminus \{z_0\}$. Fix $a \in \Omega$ and define $F : \Omega \to \mathbb{C}$ by $F(z) = \int_{\langle a,z\rangle} f$. Then F is a primitive of f and $\int_\Gamma f = 0$ for every closed contour Γ in Ω.

3.3. Consequences of Cauchy's theorem

3.3.1 Cauchy's integral formula for convex sets. Suppose Ω is a non-empty open convex set, $f : \Omega \to \mathbb{C}$ a holomorphic function on Ω, and z is a point in Ω. Then

$$g(u) = \begin{cases} \frac{f(u)-f(z)}{u-z} & \text{if } u \neq z \\ f'(z) & \text{if } u = z \end{cases}$$

satisfies the hypothesis of Cauchy's theorem for convex sets 3.2.8 so that $\int_\Gamma g = 0$ when Γ is a closed contour in Ω.

Therefore, if $z \in \Omega \setminus \Gamma^*$, then

$$f(z)\,\mathrm{Ind}_\Gamma(z) = \frac{1}{2\pi \mathrm{i}} \int_\Gamma \frac{f(u)}{u-z} du.$$

This identity is known as *Cauchy's integral formula*.

3.3.2 Holomorphic functions are analytic. The conclusions in 3.3.1 and 2.6.6 imply the central theorem of Complex Analysis.

Theorem. A holomorphic function is analytic on its domain of definition.

From now on the words holomorphic and analytic may be considered synonymous (many authors do not ever make a distinction between them).

Corollary. Any holomorphic function may be expanded into a Taylor series about any point in its domain.

3.3.3 Morera's[1] theorem. Let Ω be a non-empty open set. If $f : \Omega \to \mathbb{C}$ is continuous and if $\int_\Gamma f = 0$ for every triangular contour $\Gamma \in \Omega$, then f is holomorphic in Ω.

3.3.4 Sequences of holomorphic functions. Let Ω be a non-empty open set. Suppose $n \mapsto f_n$ is a sequence of holomorphic functions defined on Ω which converges uniformly to $f : \Omega \to \mathbb{C}$. Then f is holomorphic.

3.3.5 General integral formulas for convex sets. Suppose Ω is a non-empty open convex set and $f : \Omega \to \mathbb{C}$ a holomorphic function on Ω. If Γ is a closed contour in Ω, $z \in \Omega \setminus \Gamma^*$, and $n \in \mathbb{N}_0$, then

$$f^{(n)}(z)\,\mathrm{Ind}_\Gamma(z) = \frac{n!}{2\pi \mathrm{i}} \int_\Gamma \frac{f(u)}{(u-z)^{n+1}} du.$$

[1] Giacinto Morera (1856 – 1909)

Sketch of proof. Using 2.6.4 we may express $f^{(k)}(z_0)$ in terms of a Taylor coefficient and then compute that coefficient employing a combination of 3.3.1 and 2.6.6. □

3.3.6 Cauchy's estimate. If $f : B(z_0, R) \to \mathbb{C}$ is a holomorphic function, then the radius of convergence of its Taylor series about z_0 is at least equal to R. If $|f(z)| \le M$ whenever $z \in B(z_0, R)$, then

$$|a_n| = \frac{|f^{(n)}(z_0)|}{n!} \le \frac{M}{R^n}$$

for all $n \in \mathbb{N}_0$.

3.3.7 Liouville's[2] theorem. Every bounded entire function is constant.

3.3.8 Zeros and their order. Let z_0 be a zero of the holomorphic function f and $\sum_{n=0}^{\infty} a_n(z - z_0)^n$ the Taylor series of f about z_0. Then z_0 is an isolated point in the set of all zeros of f unless $a_n = 0$ for all $n \in \mathbb{N}_0$. If z_0 is an isolated zero of f, the number $m = \min(\{k \in \mathbb{N} : a_k \neq 0\})$ is called the *order* or the *multiplicity* of z_0 as a zero of f. In this case we have $f(z) = (z - z_0)^m g(z)$ where g is holomorphic in the domain of f and $g(z_0) \neq 0$.

3.3.9 The set of zeros of a holomorphic function. Let Ω be a non-empty open connected subset of $\mathbb{C}$ and f a holomorphic function on Ω. Let $Z(f)$ be the set of zeros of f, i.e., $Z(f) = \{a \in \Omega : f(a) = 0\}$, and A the set of limit points of $Z(f)$ in Ω. Then A and $\Omega \setminus A$ are open and, in consequence, either $Z(f) = \Omega$ or else $Z(f)$ is a set of isolated points.

3.3.10 Analytic continuation. Let f and g be holomorphic functions on Ω, a non-empty open connected subset of $\mathbb{C}$. Then the following statements hold:
(1) If $f(z) = g(z)$ for all z in a set which contains a limit point of itself, then $f = g$.
(2) If $f^{(n)}(z_0) = g^{(n)}(z_0)$ for some z_0 and all $n \in \mathbb{N}_0$, then $f = g$.

Suppose f is a holomorphic function on Ω and that Ω' is another open connected set which intersects Ω. Then there is at most one holomorphic function on $\Omega \cup \Omega'$ which coincides with f on Ω. This function (if it exists) is called the *analytic continuation* of f to $\Omega \cup \Omega'$.

3.4. The global version of Cauchy's theorem

3.4.1 Continuity of parametric integrals. Let Ω be an open set, Γ a contour in $\mathbb{C}$, and φ a continuous complex-valued function on $\Gamma^* \times \Omega$. Define $g(z) = \int_\Gamma \varphi(\cdot, z)$ on Ω. Then g is continuous on Ω.

Sketch of proof. By A.3.1 $\Gamma^* \times \overline{B}(z_0, r)$ is compact and A.2.3 gives that $\varphi|_{\Gamma^* \times \overline{B}(z_0,r)}$ is uniformly continuous so that, for all t, $|\varphi(\gamma(t), z) - \varphi(\gamma(t), z_0)| < \varepsilon$ if $|z - z_0|$ is sufficiently small. □

3.4.2 Differentiation under the integral. Let Ω, Γ, φ, and g be defined as in 3.4.1. Furthermore suppose that $\varphi(u, \cdot)$ is holomorphic in Ω for each $u \in \Gamma^*$ and denote its derivative by $\psi(u, \cdot)$, so that $\varphi(u, \cdot)$ is a primitive of $\psi(u, \cdot)$ for every fixed $u \in \Gamma^*$. If $\psi : \Gamma^* \times \Omega \to \mathbb{C}$ is continuous, then g is holomorphic in Ω.

[2]Joseph Liouville (1809 – 1882)

SKETCH OF PROOF. Note that we have $\varphi(u,z)-\varphi(u,z_0) = \int_{\langle z_0,z\rangle} \psi(u,\cdot)$ and let $h(z_0) = \int_\Gamma \psi(\cdot,z_0)$. Then

$$\left|\frac{g(z)-g(z_0)}{z-z_0} - h(z_0)\right| \leq \sum_{k=1}^{n} \int_{a_k}^{b_k} \int_0^1 |\psi(\gamma_k(t),u(s)) - \psi(\gamma_k(t),z_0)| ds |\gamma_k'(t)| dt$$

and this is less than ε times the total length of Γ. □

3.4.3 A special (but important) case. Suppose Ω is an open subset of $\mathbb{C}$ and f is a holomorphic function on Ω. Define $\varphi, \psi : \Omega \times \Omega \to \mathbb{C}$ by

$$\varphi(u,z) = \begin{cases} (f(u)-f(z))/(u-z) & \text{if } u \neq z, \\ f'(z) & \text{if } u = z \end{cases}$$

and

$$\psi(u,z) = \begin{cases} (f(u)-f(z)-f'(z)(u-z))/(u-z)^2 & \text{if } u \neq z, \\ \frac{1}{2} f''(z) & \text{if } u = z. \end{cases}$$

Then φ and ψ are continuous on $\Omega \times \Omega$ and $\psi(u,\cdot)$ is the derivative of $\varphi(u,\cdot)$.

SKETCH OF PROOF. Fix $(u_0,z_0) \in \Omega \times \Omega$. There is no difficulty if $u_0 \neq z_0$. If $u_0 = z_0$ we use $\varphi(u,z) = \int_0^1 f'((1-t)z+tu)dt$ and $\psi(u,z) = \int_0^1 t \int_0^1 f''((1-st)z+stu)dsdt$. □

3.4.4 Cauchy's integral formula, global version. Suppose Ω is an open subset of the complex plane, f is a holomorphic function on Ω, and Γ is a closed contour in Ω such that $\operatorname{Ind}_\Gamma(w) = 0$ whenever w is not an element of Ω. Then

$$f(z) \operatorname{Ind}_\Gamma(z) = \frac{1}{2\pi \mathrm{i}} \int_\Gamma \frac{f(u)}{u-z} du$$

for every $z \in \Omega \setminus \Gamma^*$.

SKETCH OF PROOF. On Ω define $g(z) = \int_\Gamma \varphi(u,z)du$ with φ as in 3.4.3. On $\Omega' = \{z \in \mathbb{C} : \operatorname{Ind}_\Gamma(z) = 0\}$ define $h(z) = \int_\Gamma f(u)/(u-z)du$. Then create an entire function by showing that $h = g$ on $\Omega \cap \Omega'$. This function must be zero. □

3.4.5 Cauchy's theorem, global version. Suppose Ω, f, and Γ satisfy the same hypotheses as in 3.4.4. Then $\int_\Gamma f = 0$.

3.4.6 Laurent[3] series. Suppose $0 \leq r_1 < r_2$ and let f be holomorphic in the annulus $\Omega = \{z \in \mathbb{C} : r_1 < |z-a| < r_2\}$. Then, f can be expressed by a *Laurent series*, i.e.,

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z-a)^n.$$

This is actually an abbreviation for

$$f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n + \sum_{n=1}^{\infty} a_{-n} (z-a)^{-n},$$

i.e., convergence of a Laurent series requires convergence of both series of positive and negative powers.

[3]Pierre Laurent (1813 – 1854)

The coefficients a_n are given by the integrals

$$a_n = \frac{1}{2\pi \mathrm{i}} \int_\gamma \frac{f(u)}{(u-a)^{n+1}} du$$

where $\gamma = a + r' \exp(\mathrm{i}t)$, $t \in [0, 2\pi]$ and $r' \in (r_1, r_2)$.

Sketch of proof. Fix $z \in \Omega$. For $j = 1, 2$ let $\gamma_j(t) = a + r'_j \exp(\mathrm{i}t)$ for $t \in [0, 2\pi]$ and suitably chosen r'_j. Then use Cauchy's integral formula for $\Gamma = \gamma_2 \ominus \gamma_1$. To find a_n for $n \geq 0$ employ 2.6.6; for the others use a variant of its proof after noting that $u - z = -(z-a)(1-(u-a)/(z-a))$. In the formula for a_n the radii r'_1 and r'_2 may be replaced by any $r' \in (r_1, r_2)$. □

The Taylor series of an analytic function is, of course, a special case of a Laurent series.

3.4.7 Cauchy's theorem and integral formula for simply connected open sets. A connected open set Ω in the complex plane is called *simply connected* if $\mathbb{C} \setminus \Omega$ has no bounded component. The set $\mathbb{C} \setminus \{0\}$ is not simply connected but $\mathbb{C} \setminus (-\infty, 0]$ is.

If Ω is a non-empty open simply connected subset of $\mathbb{C}$ and Γ a closed contour in Ω, then $\mathrm{Ind}_\Gamma(w) = 0$ whenever w is not an element of Ω. Consequently, if f is a holomorphic function on Ω, then

$$f(z)\,\mathrm{Ind}_\Gamma(z) = \frac{1}{2\pi \mathrm{i}} \int_\Gamma \frac{f(u)}{u-z} du$$

for every $z \in \Omega \setminus \Gamma^*$ and

$$\int_\Gamma f = 0.$$

CHAPTER 4

Isolated singularities

4.1. Classifying isolated singularities

Throughout this section Ω denotes an open subset of the complex plane.

4.1.1 Isolated singularities. Suppose $z_0 \in \Omega$ and $f : \Omega \setminus \{z_0\} \to \mathbb{C}$ is holomorphic. Then z_0 is called an *isolated singularity* of f.

The following are typical examples: $z \mapsto (z^2 - z_0^2)/(z - z_0)$, $z \mapsto 1/(z - z_0)$, and $z \mapsto \exp(1/(z - z_0))$.

4.1.2 Laurent series about an isolated singularity. A punctured disk $\{z \in \mathbb{C} : 0 < |z - z_0| < r\}$ is a special case of an annulus and thus a holomorphic function defined on it has a Laurent series expansion. In particular, if z_0 is an isolated singularity of f, then there is a punctured disk $\{z \in \mathbb{C} : 0 < |z - z_0| < r\}$ (possibly the punctured plane) on which

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n.$$

We call the point z_0 a *removable singularity*, if $a_n = 0$ for all $n < 0$. It is called a *pole* of *order* m or of *multiplicity* m (where $m > 0$), if $a_n = 0$ for all $n < -m$ and $a_{-m} \neq 0$. The point z_0 is called an *essential singularity*, if it is neither a pole nor a removable singularity, i.e., if the set $\{n \in \mathbb{Z} : a_n \neq 0\}$ is not bounded below.

4.1.3 Removable singularities. If z_0 is a removable singularity of a function f with a Laurent expansion on the punctured disk $B(z_0, r) \setminus \{z_0\}$, then there is an analytic continuation of f to $B(z_0, r)$. In particular, in this case f has a limit at z_0. Conversely, if z_0 is an isolated singularity of f and $\lim_{z \to z_0} (z - z_0) f(z) = 0$, then z_0 is in fact a removable singularity of f. To see this define $h(z_0) = 0$ and $h(z) = (z - z_0) f(z)$ for $z \neq z_0$. Then one may show that h is holomorphic near z_0.

4.1.4 Poles. The following statements about poles hold.

(1) The point z_0 is a pole of order m of a function f if and only if $1/f$ has an analytic continuation for which z_0 is a zero of order m.
(2) If z_0 is a pole of f and M is any positive real number, then there is a positive δ such that $|f(z)| \geq M$ for all $z \in B(z_0, \delta) \setminus \{z_0\}$.
(3) If z_0 is an isolated singularity of f and there is a natural number m such that $\lim_{z \to z_0} (z - z_0)^{m+1} f(z) = 0$, then z_0 is either a removable singularity of f or else a pole of order at most m.
(4) If f has a pole of order m at z_0, then there are complex numbers c_1, ..., c_m so that

$$z \mapsto f(z) - \sum_{k=1}^{m} \frac{c_k}{(z - z_0)^k}$$

has a removable singularity at z_0. The sum $\sum_{k=1}^m c_k/(z-z_0)^k$ is called the *principal part* or *singular part* of f at z_0.

4.1.5 Meromorphic functions. Let P be a set of isolated points in Ω without a limit point in Ω and f a holomorphic function on $\Omega \setminus P$. If no point of P is an essential singularity of f, then f is called *meromorphic* on Ω.

4.1.6 The Casorati[1]-Weierstrass theorem. Suppose f is a holomorphic function defined on $B' = B(z_0, r) \setminus \{z_0\}$. If z_0 is an essential singularity of f, then $f(B')$ is dense in $\mathbb{C}$.

Sketch of proof. Assume on the contrary that there is a $a \in \mathbb{C}$ and $\delta > 0$ such that $B(a, \delta) \cap f(B') = \emptyset$. Then consider the function $g(z) = 1/(f(z) - a)$ on B' which has a removable singularity at z_0. If z_0 is a zero of g of order $m \in \mathbb{N}$ then f has a pole of order m. If $m = 0$, then f has a removable singularity. □

4.2. The calculus of residues

4.2.1 Residues. If the holomorphic function f has the isolated singularity z_0 and the Laurent expansion $\sum_{n=-\infty}^{\infty} a_n(z - z_0)^n$, then the number

$$a_{-1} = \frac{1}{2\pi \mathrm{i}} \int_\gamma f,$$

where γ traces counterclockwise a sufficiently small circle about z_0, is called the *residue* of f at z_0 and is denoted by $\mathrm{Res}(f, z_0)$.

If $\lim_{z\to\ z_0}(z - z_0)f(z) = a \neq 0$, then z_0 is a *simple pole*, i.e., a pole of order one, of f and $a = \mathrm{Res}(f, z_0)$.

4.2.2 The residue theorem. Suppose Ω is an open subset of the complex plane, f is a holomorphic function on $\Omega' = \Omega \setminus \{z_1, ..., z_n\}$, and Γ is a closed contour in Ω' such that $\mathrm{Ind}_\Gamma(w) = 0$ whenever w is not an element of Ω. Then

$$\sum_{k=1}^{n} \mathrm{Res}(f, z_k) \,\mathrm{Ind}_\Gamma(z_k) = \frac{1}{2\pi \mathrm{i}} \int_\Gamma f.$$

Sketch of proof. Let $m_k = \mathrm{Ind}_\Gamma(z_k)$, $\gamma_k(t) = z_k + r_k \exp(\mathrm{i} m_k t)$ for $t \in [0, 2\pi]$ and sufficiently small but positive r_k, and $\Gamma' = \bigoplus_{k=1}^n \gamma_k$. Then $\int_\Gamma f = \int_{\Gamma'} f$. □

4.2.3 Some examples.

$$\int_{-\infty}^{\infty} \frac{1}{1 + x^4} dx$$

$$\int_{-\infty}^{\infty} \frac{1}{\exp(x) + \exp(-x)} dx$$

$$\int_0^{2\pi} \frac{1}{2 + \sin x} dx$$

The following hints might be useful: For the first integral consider a contour made up from the interval $[-R, R]$ and a semi-circle of radius R. For the second one consider a rectangle with vertices $\pm R$ and $\pm R + \pi \mathrm{i}$. Finally, for the last integral let $u = \exp(\mathrm{i}t)$ and note that $2\mathrm{i}\sin(t) = u - 1/u$.

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[1] Felice Casorati (1835 – 1890)

4.2.4 Counting zeros and poles. Suppose f is a non-zero meromorphic function on the open set Ω, Z is the set of its zeros, and P is the set of its poles. For any point $z_0 \in \Omega$ there is a unique integer $M(z_0)$ denoting the smallest index for which the coefficient in the Laurent expansion of f about z_0 is non-zero. In fact, if z_0 is a zero or a pole of f its multiplicity is $M(z_0)$ or $-M(z_0)$, respectively.

Let Γ be a closed contour in $\Omega\backslash(Z\cup P)$ such that $\operatorname{Ind}_\Gamma(w) = 0$ whenever $w \in \mathbb{C}\backslash\Omega$. There are at most finitely many zeros and poles in those components of Ω which have a non-zero index. To see this note that $A \cup \Gamma^*$ is a compact subset of Ω if $A = \{z \in \mathbb{C} : \operatorname{Ind}_\Gamma(z) \neq 0\}$. It follows that

$$\frac{1}{2\pi \mathrm{i}} \int_\Gamma \frac{f'}{f} = \sum_{z\in Z\cup P} M(z) \operatorname{Ind}_\Gamma(z).$$

In particular, if f is holomorphic on Ω and $\operatorname{Ind}_\Gamma$ assumes only the values 0 and 1, then $\frac{1}{2\pi \mathrm{i}} \int_\Gamma \frac{f'}{f}$ is the number of zeros (counted according to their multiplicities) in the set $\{z \in \mathbb{C} : \operatorname{Ind}_\Gamma(z) = 1\}$.

4.2.5 Rouché's[2] theorem. Let f and g be meromorphic functions on the open set Ω and assume that $\overline{B}(a, r) \subset \Omega$. Let $n_z(f)$ be the number of zeros of f in $B(a, r)$, counted according to their multiplicity. Similarly, $n_p(f)$ is the count of poles, and $n_z(g)$ and $n_p(g)$ are the analogous quantities for g. If no zero or pole lies on the circle $C = \{z : |z - a| = r\}$ and if $|f(z) - g(z)| < |f(z)| + |g(z)|$ for all $z \in C$, then $n_z(f) - n_p(f) = n_z(g) - n_p(g)$.

Sketch of proof. Let γ parameterize the circle and set $h = f/g$. Then $\int_\gamma h'/h = 2\pi \mathrm{i} \operatorname{Ind}_{h\circ\gamma}(0) = 0$ since $(h \circ \gamma)^* \cap (-\infty, 0] = \emptyset$. □

4.2.6 Behavior of a holomorphic functions near a zero. Suppose $f : \Omega \to \mathbb{C}$ is holomorphic and z_0 is a zero of order m of f. Then there are positive numbers r and R such that the following statement is true. If $|w| < R$, then $f - w$ has precisely m zeros (counting multiplicities) in $B(z_0, r)$.

Sketch of proof. Note that $f(z) = a(z - z_0)^m h(z)$ where h is holomorphic on Ω, $h(z_0) = 1$, and $a \neq 0$. Since $h - 1$ is bounded near z_0 and vanishes at z_0, there is an $r > 0$ such that $|h(z) - 1| \leq 1/2$ for $|z - z_0| \leq r$. Now let $F(z) = a(z - z_0)^m h(z) - w$ and $G(z) = a(z - z_0)^m - w$. If $|z - z_0| = r$ and $|w| < R = |a|r^m/2$, we have $|F(z) - G(z)| \leq |a|r^m/2 < |a(z - z_0)^m - w| \leq |G(z)| + |F(z)|$. Now apply Rouché's theorem and recall that w/a has precisely m m-th roots. □

4.2.7 The open mapping theorem. Suppose f is a non-constant holomorphic function on the open and connected set Ω. Then $f(\Omega')$ is open whenever Ω' is an open subset of Ω.

Sketch of proof. Suppose $f(z_0) = w_0 \in f(\Omega')$. Let $F = f - w_0$ and $G = f - w$. Then $|F(z_0 + r \exp(\mathrm{i}t))| \geq \delta > 0$ for some $r > 0$ and all $t \in [0, 2\pi]$. The proof follows once we determine how close w has to be to w_0 so that Rouché's theorem 4.2.5 applies to F and G. □

4.2.8 The maximum modulus theorem. Suppose f is a holomorphic function on the open and connected set Ω and $|f(z_0)|$ is a local maximum of $|f|$. Then f is constant.

[2]Eugène Rouché (1832 – 1910)

CHAPTER 5

A zoo of functions

In this chapter we investigate briefly the most elementary functions of analysis. Polynomials and exponential functions have been introduced earlier since they are too important to postpone their use.

5.1. Polynomial functions

5.1.1 Polynomial functions. If n is a non-negative integer and $a_0, a_1, \ldots, a_n$ are complex numbers, then the function $p : \mathbb{C} \to \mathbb{C}$ defined by

$$p(z) = \sum_{k=0}^{n} a_k z^k$$

is called a *polynomial* (of a single variable). The integer n is called the *degree* of p if a_n is different from 0. It is then called the *leading coefficient* of p. The zero function is also a polynomial but no degree is assigned to it. Any polynomial is an entire function.

If p and q are polynomials of degree n and k, respectively, then $p + q$ and pq are also polynomials. The degree of $p+q$ is the larger of the numbers n and k unless $n = k$ in which case the degree is at most n. The degree of pq equals $n + k$.

5.1.2 The fundamental theorem of algebra. Suppose p is a polynomial of degree $n \geq 1$. Then there exist numbers a and z_1, ..., z_n (not necessarily distinct) such that

$$p(z) = a \prod_{k=1}^{n} (z - z_k).$$

To prove this use either Rouchés theorem or the fact that $1/p$ would be entire if p had no zeros.

5.1.3 Zeros of polynomials. The coefficients a_k of a monic polynomial are given as symmetric polynomials in terms of the zeros. As a matter of fact,

$$a_{n-k} = \sum_{1 \leq j_1 < \ldots < j_k \leq n} (-1)^k z_{j_1} \ldots z_{j_k}.$$

In particular, $a_{n-1} = -\sum_{k=1}^{n} z_k$ and $a_0 = (-1)^n \prod_{k=1}^{n} z_k$. They vary continuously with the zeros.

That the converse is also true can be proved with the aid of Rouché's theorem. The precise statement is as follows: Suppose $f(z) = \sum_{k=0}^{n} a_k z^k$ and $g(z) = \sum_{k=0}^{n} b_k z^k$ are two monic polynomials of degree at most n. If z_0 is the only zero of f in $\overline{B}(z_0, r)$ and if the multiplicity of z_0 is m then the following holds: For every $\varepsilon \in (0, r)$ there is a $\delta > 0$ such that, if $|a_k - b_k| < \delta$ for $k = 0, ..., n$, then g has precisely m zeros in $B(z_0, \varepsilon)$ (counting multiplicities).

5.1.4 Among the entire functions only polynomials grow like powers. Suppose f is an entire function and $k \mapsto r_k$ is an increasing unbounded sequence of positive numbers. Furthermore, suppose there are positive real numbers N and C such that $|f(z)| \leq C|z|^N$ for all z lying on any of the circles $|z| = r_k$. Then f is a polynomial whose degree is less then or equal to N.

It is in fact enough to require $\mathrm{Re}(f(z)) \leq C|z|^N$ to arrive at the same conclusion. To see this use the Laurent expansion 3.4.6 of f to obtain $2\pi r_k^n a_n = \int_0^{2\pi} f(\gamma_k(t)) \exp(-\mathrm{i}nt) dt$ for $n \in \mathbb{Z}$. Therefore, using that $a_n = 0$ if $n < 0$, we get $\pi r_k^n a_n = \int_0^{2\pi} \mathrm{Re}\, f(\gamma_k(t)) \exp(-\mathrm{i}nt) dt$ for $n \in \mathbb{N}$ and $2\pi \,\mathrm{Re}\, a_0 = \int_0^{2\pi} \mathrm{Re}\, f(\gamma_k(t)) dt$. We also have $\int_0^{2\pi} C r_k^N \exp(-\mathrm{i}nt) dt = 0$ so that $\pi r_k^n |a_n| \leq \int_0^{2\pi} (C r_k^N - \mathrm{Re}\, f(\gamma_k(t))) dt = 2\pi C r_k^N - 2\pi \,\mathrm{Re}\, a_0$. Hence $a_n = 0$ when $n > N$.

Thus the real (or the imaginary part) of a polynomial grows roughly at the same pace as the modulus. This is, in fact, a special case of a more general result, the Borel-Carathéodory[1] theorem, which allows to estimate the modulus of an entire function by its real (or imaginary) part.

5.2. Rational functions

5.2.1 The extended complex plane. We define the *extended complex plane* $\mathbb{C}_\infty$ as the union of $\mathbb{C}$ and another point, not in $\mathbb{C}$, which we denote by ∞. We call a subset of $\mathbb{C}_\infty$ open, if it is either an open subset of $\mathbb{C}$ or else the union of $\{\infty\}$ and the complement of a compact subset of $\mathbb{C}$. The set of all open subsets of $\mathbb{C}_\infty$ is a topology for $\mathbb{C}_\infty$.

Under this topology $\mathbb{C}_\infty$ is a compact space sometimes called the *one-point compactification* of $\mathbb{C}$.

5.2.2 The Riemann sphere. The set $S^2 = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1\}$ is called the *unit sphere*. The point $(0, 0, 1)$ is called the *north pole* of the sphere. Let (x, y, z) be a point in S^2 other than the north pole. The map $t \mapsto (tx, ty, tz + 1 - t)$, $t \in \mathbb{R}$, describes a straight line passing through the north pole and the point (x, y, z). This line intersects the equator plane (i.e., the plane where the third component is 0) in the unique point $(x/(1 - z), y/(1 - z), 0)$. We have thus defined the map

$$\pi : S^2 \setminus \{(0, 0, 1)\} \to \mathbb{C} : (x, y, z) \mapsto \frac{x + iy}{1 - z}.$$

This map, called the *stereographic projection*, is bijective. Points in the southern hemisphere correspond to points in the unit disk, points on the equator correspond to points on the unit circle, and points in the northern hemisphere correspond to points outside the closed unit disk.

With this interpretation the sphere S^2 is called the *Riemann sphere*. Note that points on the sphere close to the north pole are those whose image points in the plane have large absolute values. The Riemann sphere is homeomorphic to the extended complex plane $\mathbb{C}_\infty$.

5.2.3 Continuity and the extended complex plane. We can view a meromorphic function on Ω as a function from Ω to $\mathbb{C}_\infty$ by defining

$$\tilde{f}(u) = \begin{cases} f(u) & \text{if } u \text{ is not a pole} \\ \infty & \text{if } u \text{ is a pole.} \end{cases}$$

Note that $\tilde{f}$ is continuous at every point of Ω, in the sense of A.1.2, even the poles.

[1] Constantin Carathéodory (1873 – 1950)

5.2.4 Rational functions. Let p and q be polynomials and $Q = \{z \in \mathbb{C} : q(z) = 0\}$. Assume $Q \neq \mathbb{C}$, i.e., q is not the zero polynomial. Then, the function $r : (\mathbb{C} \setminus Q) \to \mathbb{C}$ given by

$$r(z) = \frac{p(z)}{q(z)}$$

is called a *rational function*.

As a meromorphic function on $\mathbb{C}$ a rational function may also be interpreted as a continuous function from $\mathbb{C}$ to $\mathbb{C}_\infty$. We can even extend the domain of r to $\mathbb{C}_\infty$ and define $r(\infty) = \lim_{z\to\infty} r(z)$ (which may again be infinity).

5.2.5 Möbius transforms. A rational function of the type $z \mapsto (az + b)/(cz + d)$ where $ad - bc \neq 0$ is called a *Möbius*[2] *transform*. The set of all Möbius transforms forms a group under composition. Special cases of Möbius transforms are the translations $z \mapsto z + b$, the dilations and rotations $z \mapsto az$ where $a > 0$ and $|a| = 1$, respectively, and the inversion $z \mapsto 1/z$. Any Möbius transform is a composition of at most five of these.

A Möbius transform may be interpreted as a homeomorphism from the Riemann sphere to itself. Conversely, an injective meromorphic function on $\mathbb{C}$ is a Möbius transform.

Möbius transforms have many interesting properties to which an entire chapter might be devoted.

5.3. Exponential and trigonometric functions

5.3.1 The exponential function. We defined the exponential function already in 2.6.5. Hence most of the following will be a repetition of results obtained there. The function $\exp : \mathbb{C} \to \mathbb{C}$ defined by

$$\exp(z) = \sum_{k=0}^{\infty} \frac{z^k}{k!}$$

is called the *exponential function*. It is an entire function and $\exp'(z) = \exp(z)$.

If $x, y \in \mathbb{C}$, then $\exp(x + y) = \exp(x)\exp(y)$. In particular, $\exp(z)$ is never zero and $\exp(-z) = 1/\exp(z)$.

The exponential function has period $2\pi\mathrm{i}$, i.e., $\exp(z + 2\pi\mathrm{i}) = \exp(z)$. If p is any period of exp, then there is an integer m such that $p = 2m\pi i$. Let a be a fixed real number and $S_a = \{z \in \mathbb{C} : a < \operatorname{Im}(z) \leq a + 2\pi\}$. Then $\exp|_{S_a}$ maps S_a bijectively to $\mathbb{C} \setminus \{0\}$.

5.3.2 Trigonometric functions. The trigonometric functions are defined by

$$\cos(z) = \frac{\exp(\mathrm{i}z) + \exp(-\mathrm{i}z)}{2},$$

and

$$\sin(z) = \frac{\exp(\mathrm{i}z) - \exp(-\mathrm{i}z)}{2\mathrm{i}}.$$

They are entire functions, called the *cosine* and *sine function*, respectively.

The following properties, familiar from the real case, extend to the complex case:

(1) Derivatives: $\sin'(z) = \cos(z)$ and $\cos'(z) = -\sin(z)$.
(2) The Pythagorean theorem: $(\sin z)^2 + (\cos z)^2 = 1$ for all $z \in \mathbb{C}$.

[2] August Ferdinand Möbius (1790 – 1868)

(3) Addition theorems:

$$\sin(z+w) = \sin(z)\cos(w) + \cos(z)\sin(w)$$

and

$$\cos(z+w) = \cos(z)\cos(w) - \sin(z)\sin(w)$$

for all $z, w \in \mathbb{C}$.

(4) Taylor series:

$$\sin(z) = \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n+1}}{(2n+1)!}$$

and

$$\cos(z) = \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n}}{(2n)!}.$$

In particular, sin and cos are extensions of the functions defined in Real Analysis.

(5) $\cos(z) = \sin(z + \pi/2)$.

(6) $\sin(z) = 0$ if and only z is an integer multiple of π.

(7) The range of both, the sine and the cosine function, is $\mathbb{C}$.

5.4. The logarithmic function and powers

5.4.1 The real logarithm. Since $\exp : \mathbb{R} \to (0, \infty)$ is a bijective function, it has an inverse $\ln : (0, \infty) \to \mathbb{R}$. One shows that $\ln(1) = 0$ and $\ln'(x) = 1/x$ so that, by the fundamental theorem of calculus

$$\ln(x) = \int_1^x \frac{dt}{t}$$

for all $x > 0$.

5.4.2 The logarithm. Suppose Ω is a simply connected non-empty open subset of $\mathbb{C} \setminus \{0\}$ and Γ is a contour in $\mathbb{C}\setminus\{0\}$ connecting 1 to $z_0 \in \Omega$. Then we define the function $L_\Gamma : \Omega \to \mathbb{C}$ by setting

$$L_\Gamma(z) = \int_{\Gamma \oplus \beta} \frac{du}{u}$$

where β is a contour in Ω connecting z_0 to z. As the notation suggests $L_\Gamma(z)$ does not depend on β as long as β remains in Ω. The function L_Γ is called a *branch of the logarithm* on Ω.

Suppose now that Γ' is another contour in $\mathbb{C} \setminus \{0\}$ connecting 1 to a point $z_0' \in \Omega$. If γ is a contour in Ω connecting z_0 and z_0', then

$$L_\Gamma(z) - L_{\Gamma'}(z) = 2\pi \mathrm{i}\, \mathrm{Ind}_{\Gamma\oplus\gamma\ominus\Gamma'}(0).$$

Thus there are (at most) countably many different functions L_Γ which are defined this way, even though there are many more contours Γ connecting 1 to a point in Ω. In fact, for each $m \in \mathbb{Z}$ there is a Γ' such that $L_\Gamma(z) - L_{\Gamma'}(z) = 2m\pi i$. This shows that defining the logarithm as an antiderivative comes with certain difficulties (to say the least). At the same time this behavior gives rise to a lot of interesting mathematics.

L_Γ is a holomorphic function on Ω with derivative $1/z$. Moreover, $\exp(L_\Gamma(z)) = z$ for all $z \in \Omega$ but $L_\Gamma(\exp(z))$ may well differ from z by an integer multiple of $2\pi\mathrm{i}$.

5.4.3 The principal branch of the logarithm. Let $\Omega = \mathbb{C} \setminus (-\infty, 0]$. Then

$$\log(z) = \int_\gamma \frac{du}{u},$$

where γ is contour in Ω connecting 1 and z, is uniquely defined for any $z \in \Omega$ (i.e., $\log = L_\Gamma$ where $\Gamma : [0,1] \to \mathbb{C} : t \mapsto 1$). It is called the *principal branch* of the *logarithm*. If $z \in \Omega$ has polar representation $z = r\exp(it)$ where $t \in (-\pi, \pi)$ and $r > 0$, then

$$\log(z) = \ln(r) + it.$$

The range of log is the strip $\{z \in \mathbb{C} : |\operatorname{Im}(z)| < \pi\}$. In particular, $\log(\exp(z)) = z$ if and only if $|\operatorname{Im}(z)| < \pi$.

The Taylor series of $z \mapsto \log(1+z)$ about $z = 0$ has radius of convergence 1 and is given by

$$\log(1+z) = -\sum_{n=1}^{\infty} \frac{(-z)^n}{n}.$$

If $\operatorname{Re}(a)$ and $\operatorname{Re}(b)$ are positive, then $\log(ab) = \log(a) + \log(b)$. The conclusion may be wrong when the hypothesis is not satisfied.

5.4.4 Powers. Suppose a is a non-zero complex number and L is a branch of the logarithm whose domain includes a. If $b \in \mathbb{Z}$, then $a^b = \exp(bL(a))$. We may therefore define

$$a^b = \exp(bL(a))$$

for any $b \in \mathbb{C}$. Be aware, though, that, in general, the value of a^b depends on the branch of the logarithm chosen and is therefore ambiguous when $b \notin \mathbb{Z}$. In most cases one chooses, of course, the principal branch of the logarithm and defines $a^b = \exp(b\log(a))$ (forsaking the definition of powers of negative numbers). In this case we get $\mathrm{e}^z = \exp(z)$ for all $z \in \mathbb{C}$, after we define $\mathrm{e} = \exp(1)$.

5.4.5 Power functions. Suppose Ω is a simply connected non-empty open subset of $\mathbb{C} \setminus \{0\}$. Let p be a complex number and $L : \Omega \to \mathbb{C}$ a branch of the logarithm. The function $\Omega \to \mathbb{C} : z \mapsto z^p = \exp(pL(z))$ is called a *branch of the power function*. Each such branch is a holomorphic function which never vanishes. Note that $1/z^p = z^{-p}$.

If $p \in \mathbb{N}_0$ all branches of the associated power function may be analytically extended to the entire complex plane and any branch gives then rise to one and the same function. The same is true if $p \in \mathbb{Z}$ is negative, except that the function may not be extended to 0 since 0 is then a pole of the function.

If $p \in \mathbb{Q}$ there are only finitely many different branches of $z \mapsto z^p$ in any simply connected open set not containing 0. In fact, if m/n is a representation of p in lowest terms, there will be n branches of $z \mapsto z^p$.

5.4.6 Holomorphic functions without zeros. Suppose f is a holomorphic function defined on a simply connected open set Ω. If f has no zeros, then f'/f has an antiderivative $\tilde{g}$ and $f\exp(-\tilde{g})$ is constant. It follows that there is an holomorphic function g on Ω such that $f(z) = \exp(g(z))$.

5.4.7 A definite integral. We will later use the following result (which is also an instructive example):

$$\int_0^{2\pi} \log|1 - \exp(it)|dt = 0.$$

To see this let $f(z) = z^{-1}\log(1-z)$ and note that 0 is a removable singularity of f. Hence we get for small positive δ that $\int_{\gamma_1 \ominus \gamma_2} f = 0$ when $\gamma_1(t) = \exp(\mathrm{i}t)$, $t \in [\delta, 2\pi - \delta]$ and $\gamma_2(t) = (1-t)\exp(\mathrm{i}\delta) + t\exp(-\mathrm{i}\delta)$, $t \in [0,1]$. Since $\mathrm{Re}(\log(1-z)) = \ln|1-z|$, $|1-\mathrm{e}^{it}| \geq |t|/2$ for sufficiently small t, $|1-\gamma_2(t)| \geq |1-\cos\delta| \geq \delta^2/4$, and $t\ln t - t$ is an antiderivative of $\ln t$, we obtain the claim upon taking the limit $\delta \to 0$.

CHAPTER 6

Entire functions

6.1. Infinite products

6.1.1 Infinite products. If $n \mapsto z_n$ is a sequence of complex numbers, we denote the sequence $k \mapsto p_k = \prod_{n=1}^{k} z_n$ of partial products by $\prod_{n=1}^{\infty} z_n$. If $\lim_{k\to\infty} p_k$ exists, we will call it the *infinite product* of the numbers z_n and also denote it by $\prod_{n=1}^{\infty} z_n$ (abusing notation as we do for series).

6.1.2 Convergence criteria for infinite products. Suppose $\sum_{n=1}^{\infty}(z_n - 1)$ is absolutely convergent. Then there is an $n_0 \in \mathbb{N}$ such that $|z_n - 1| < 1/2$ for all $n \geq n_0$. Moreover, $\sum_{n=n_0}^{\infty} \log(z_n)$ is absolutely convergent and $\prod_{n=n_0}^{\infty} z_n$ converges to a non-zero number. The latter limit is independent of the order of the factors since

$$\prod_{n=n_0}^{\infty} z_n = \exp\Big(\sum_{n=n_0}^{\infty} \log(z_n)\Big).$$

We will then say that $\prod_{n=1}^{\infty} z_n$ *converges absolutely*.

If $\prod_{n=1}^{\infty} z_n$ converges absolutely to 0, then finitely many and only finitely many of the z_n vanish.

6.1.3 Infinite products of analytic functions. Let Ω be a non-empty open set and $n \mapsto f_n$ a sequence of holomorphic functions on Ω none of which is identically equal to zero. If $\sum_{n=1}^{\infty} |f_n - 1|$ converges uniformly on Ω, then $\prod_{n=1}^{\infty} f_n$ converges to a holomorphic function f on Ω. If z_0 is a zero of f, then it is a zero of only finitely many of the functions f_n and the order of z_0 as a zero of f is the sum of the orders of z_0 as a zero of f_n.

6.2. Weierstrass's factorization theorem

6.2.1 Elementary factors. The functions

$$E_0(z) = 1 - z$$
$$E_p(z) = (1 - z)\exp(z + \ldots + z^p/p)$$

are called *elementary factors*. Note that $E_p(\cdot/a)$ is entire and has a simple zero at a but no other zero.

If $|z| \leq 1/2$ we have the following estimates:

$$|E_p(z) - 1| \leq 2\mathrm{e}|z|^{p+1}$$

and

$$-2\mathrm{e}|z|^{p+1} \leq \log(|E_p(z)|) \leq 2\mathrm{e}|z|^{p+1}.$$

6.2.2 Prescribing zeros of an entire function. Suppose a_n is a (finite or infinite) sequence of non-zero complex numbers with no finite limit point. Then there is a sequence

$n \mapsto p_n \in \mathbb{N}_0$, e.g., $p_n = n$, such that

$$\forall r > 0 : \sum_n (r/|a_n|)^{p_n+1} < \infty.$$

Moreover,

$$\prod_n E_{p_n}(z/a_n)$$

converges to an entire function f. The zeros of f are precisely the numbers a_n and the order of z_0 as a zero of f equals $\#\{n : z_0 = a_n\}$.

6.2.3 The Weierstrass factorization theorem. Let f be an entire function and suppose that a_n, $n \in \mathbb{N}$, are the non-zero zeros of f repeated according to their multiplicity. Then there is an integer m, an entire function g, and a sequence $n \mapsto p_n \in \mathbb{N}_0$ such that

$$f(z) = z^m \exp(g(z)) \prod_n E_{p_n}(z/a_n).$$

6.3. Counting zeros using Jensen's theorem

6.3.1 Order of an entire function. An entire function is said to be of finite order if the set

$$S = \{a \in [0, \infty) : \exists r_0 > 0 : |z| > r_0 \Longrightarrow |f(z)| \leq \exp(|z|^a)\}$$

is not empty and otherwise of infinite order. In the former case the number $\inf S$ is called the *order* of f.

6.3.2 Exponent of convergence. Let $n \mapsto a_n$ be a sequence of non-zero complex numbers. Then

$$\tau = \inf\{s > 0 : \sum_{n=1}^{\infty} |a_n|^{-s} < \infty\},$$

if it exists, is called the *exponent of convergence* of the sequence $n \mapsto a_n$. Note that, if τ is finite, the sequence $n \mapsto a_n$ has no finite limit point.

6.3.3 Counting zeros. If f is an entire function we define $n_f(r)$ to be the number of zeros, counting multiplicities, of f in $\overline{B}(0, r)$.

6.3.4 Jensen's[1] theorem. Suppose f is an entire function and $f(0) \neq 0$. Denote its non-zero zeros, repeated according to their multiplicity and ordered by size, by a_k, $k \in N$. Then

$$\sum_{k=1}^{n(r)} \log(r/|a_k|) + \log|f(0)| = \frac{1}{2\pi} \int_0^{2\pi} \log|f(r \exp(\mathrm{i}t))| dt.$$

6.3.5 The growth of an entire function controls the number of its zeros. Let f be an entire function of finite order ρ for which $f(0) = 1$ and let ε be a positive number. Then there is an $R > 0$ such that

$$n_f(r) \leq (\mathrm{e}r)^{\rho+\varepsilon}$$

for all $r \geq R$.

[1] Johan Jensen (1859 – 1925)

6.3.6 Relating growth order and exponent of convergence. Suppose f is an entire function of finite order ρ. Then the exponent of convergence τ of the sequence of non-zero zeros of f is not more than ρ.

Weierstrass's factorization theorem takes the form

$$f(z) = z^m \exp(g(z)) \prod_n E_p(z/a_n)$$

where $p \in \mathbb{N}_0$ satisfies $p \leq \tau < p + 1$.

SKETCH OF PROOF. We may assume that $f(0) = 1$ and that the non-zero zeros of f are labeled by a_n which are repeated according to their multiplicity and ordered by absolute value. Thus, if $s > \rho$ and $0 < \varepsilon < s - \rho$, we obtain from 6.3.5

$$|a_n|^{-s} \leq \mathrm{e}^s n_f(|a_n|)^{-s/(\rho+\varepsilon)} \leq \mathrm{e}^s n^{-s/(\rho+\varepsilon)}$$

when $|a_n| = r > R$. □

6.4. Hadamard's factorization theorem

6.4.1 Canonical products. Let $n \mapsto a_n$ be a sequence of non-zero complex numbers with finite exponent of convergence τ. If $p \in \mathbb{N}_0$ is such that $p \leq \tau < p + 1$ then $k(z) = \prod_n E_p(z/a_n)$ is called the *canonical product* for the sequence $n \mapsto a_n$.

The function k is an entire function of order τ.

6.4.2 Controlling the decay of canonical products. Suppose a_n is a (finite or infinite) sequence of non-zero complex numbers with a finite exponent of convergence τ. Let $p \in \mathbb{N}_0$ be such that $p \leq \tau < p + 1$. If $\tau < s \leq p + 1$ and $|z - a_n| \geq |a_n|^{-p-1}$ for all $n \in N$, then

$$\prod_n |E_p(z/a_n)| \geq \exp(-c|z|^s)$$

for some $c > 0$.

Note that the hypothesis $|z - a_n| \geq |a_n|^{-p-1}$ can be achieved for all z on a sequence of sufficiently large circles of radius r_k where $k < r_k < k + 1$ and all $n \in \mathbb{N}$.

6.4.3 Hadamard's[2] Factorization Theorem. Let f be an entire function of order ρ with non-zero zeros (repeated according to their multiplicities) a_n, $n \in N$. Let τ be the exponent of convergence of the sequence $n \mapsto a_n$ and define non-negative integers p and q by requiring $p \leq \tau < p + 1$ and $q \leq \rho < q + 1$. Also let $m \in \mathbb{N}_0$ be the order of 0 as a zero of f. Then there exists a polynomial g of degree at most q such that

$$f(z) = z^m \exp(g(z)) \prod_{n \in N} E_p(z/a_n).$$

6.4.4 The very little Picard theorem. An entire function of finite order that misses two values is constant.

To prove this assume that the entire function f of finite order misses the values α and β, where $\alpha \neq \beta$. Then $h = (f - \alpha)/(\beta - \alpha)$ is entire, of finite growth order, and misses the values 0 and 1. Hadamard's theorem 6.4.3 shows that $h = \exp(g)$ for some polynomial g. Since g has no zero it must be constant by the fundamental theorem of algebra, Theorem 5.1.2.

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[2]Jacques Hadamard (1865 – 1963)

Picard's[3] little theorem states that this so for every entire function. Picard's great theorem states that near an isolated singularity a holomorphic function takes every value, with at most one exception, infinitely often; this is an extension of the Casorati-Weierstrass theorem.

6.4.5 An entire function of finite non-integral order assumes every value infinitely often. If $f - \alpha$ has no zeros it must be of integral order according to Hadamard's theorem 6.4.3. The claim follows now since a canonical product with finitely many zeros has order 0.

[3]Émile Picard (1856 – 1941)

APPENDIX A

Topology in metric spaces

The following material is covered in a course on topology.

A.1. Basics

A.1.1 Topology for a set. Let X be a set. The system τ of subsets of X is called a *topology* for X if it has the following properties:

(1) $\emptyset$ and X are in τ
(2) Any union of elements of τ is again in τ.
(3) The intersection of two elements of τ is again in τ.

If τ is a topology for X, the pair (X, τ) is called a *topological space*. The elements of τ are called *open* sets, while their complements are called *closed* sets.

A set V is called a *neighborhood* of $x \in X$, if it is open and contains x.

A.1.2 Continuity. Suppose X and Y are topological spaces and f is a function from X to Y. Then f is called *continuous* at $x \in X$, if for every neighborhood V of $f(x)$ there is a neighborhood U of x such that $f(U) \subset V$. The function $f : X \to Y$ is called *continuous*, if it is continuous at every point $x \in X$. Equivalently, f is continuous, if the pre-image of any open subset of Y is an open subset of X.

A.1.3 Metric spaces. Let X be a set. A function $d : X \times X \to [0, \infty)$ is called a *metric* or a *distance function* in X if it has the following properties:

(1) $d(x_1, x_2) = 0$ if and only if $x_1 = x_2$.
(2) $d(x_1, x_2) = d(x_2, x_1)$ for all $x_1, x_2 \in X$.
(3) $d(x_1, x_3) \leq d(x_1, x_2) + d(x_2, x_3)$ for all $x_1, x_2, x_3 \in X$ (*triangle inequality*).

(X, d) is then called a *metric space* and the number $d(x_1, x_2)$ is called the *distance* between x_1 and x_2.

If X' is a subset of X then (X', d') is again a metric space when d' is the restriction of d to $X' \times X'$.

If $\mathbb{R}^n$ is equipped with the distance function d defined by $d(x, y)^2 = \sum_{k=1}^{n} (x_k - y_k)^2$, it becomes a metric space called the Euclidean space.

A.1.4 Open and closed balls. Let (X, d) be a metric space, x_0 a point in X, and r a non-negative real number. Then the set $B(x_0, r) = \{x \in X : d(x, x_0) < r\}$ is called the open *ball* of radius r centered at x_0 while $\overline{B}(x_0, r) = \{x \in X : d(x, x_0) \leq r\}$ is called the closed ball of radius r centered at x_0. Note that $B(x_0, 0) = \emptyset$ so that the empty set is an open ball. Similarly, the singleton $\{x_0\} = \overline{B}(x_0, 0)$ is a closed ball.

A subset of a metric space is called *bounded* if it is contained in some ball.

A.1.5 Metric spaces are topological spaces. Let (X, d) be a metric space. Then

$$\tau = \{U \subset X : U \text{ is a union of open balls}\}$$

is a topology for X. This follows easily once one proves with the help of the triangle inequality that the intersection of two open balls is a union of open balls and hence an open set.

A.1.6 Sequences and their limits. Let (X, d) be a metric space. A function $x : \mathbb{N} \to X : n \mapsto x_n$ is called a *sequence* in X. We say that the sequence x is *convergent* if the following statement holds:

$$\exists L \in X : \forall \varepsilon > 0 : \exists N > 0 : \forall n > N : d(x_n, L) < \varepsilon.$$

If a sequence is not convergent, we say that it *diverges* or is *divergent*.

If a sequence x is convergent there is only one element in X to which it converges. This element is called the *limit* of the sequence x and is denoted by $\lim_{n\to\infty} x_n$.

A.1.7 Continuity. Let (X, d) and (Y, ρ) be metric spaces and f a function from X to Y. The definition of continuity given in A.1.2 translates in the context of metric spaces as follows. The function f is called *continuous* at $a \in X$ if the following statement is true:

$$\forall \varepsilon > 0 : \exists \delta > 0 : \forall x \in X : d(x, a) < \delta \Rightarrow \rho(f(x), f(a)) < \varepsilon.$$

A.2. Compactness

A.2.1 Compact sets. A subset S of a topological space is called *compact* if every *open cover* of S has a *finite subcover*. More precisely, whenever $\mathcal{V}$ is a family of open sets such that $S \subset \bigcup_{V\in\mathcal{V}} V$ then there is a finite subset $\{V_1, ..., V_n\}$ of $\mathcal{V}$ such that $S \subset \bigcup_{k=1}^n V_k$.

A closed subset of a compact set is compact.

A.2.2 Images of compact sets under continuous functions are compact.

A.2.3 Uniform continuity. Let (X, d) and (Y, ρ) be metric spaces and f a function from X to Y. Then f is called uniformly continuous if the following statement holds:

$$\forall \varepsilon > 0 : \exists \delta > 0 : \forall x, x' \in X : d(x, x') < \delta \Rightarrow \rho(f(x), f(x')) < \varepsilon.$$

If X is compact and $f : X \to Y$ is continuous, then it is uniformly continuous.

A.2.4 Sequences in compact metric spaces have convergent subsequences.

A.2.5 The Heine[1]-Borel[2] theorem. The Heine-Borel theorem states that a subset of $\mathbb{R}^n$ is compact if and only if it is closed and bounded. It is a very important result since it makes it easy to check compactness which in turn is an essential ingredient for several important results.

A.2.6 The Bolzano[3]-Weierstrass theorem. Suppose X is a metric space. Every sequence in X has a convergent subsequence if and only if X is compact.

A.3. Product spaces

A.3.1 Product spaces. Let (X, d) and (X', d') be metric spaces. The function

$$\rho : (X \times X') \times (X \times X') \to [0, \infty) : ((x, x'), (y, y')) \mapsto \max\{d(x, y), d'(x', y')\}$$

is a distance function in $X \times X'$ turning it into a metric space.

Theorem. The cartesian product of compact metric spaces is compact.

[1] Eduard Heine (1821 – 1881)
[2] Émile Borel (1871 – 1956)
[3] Bernard Bolzano (1781 – 1848)

Glossary

annulus: In a metric space an annulus is a set lying between two concentric circles.

binary operation: A binary operation on a set A is a function from $A \times A$ to A. It is customary to express a binary operation as $a \star b$ (or with other symbols in place of $\star$).

circumference of a triangle: The sum of the edge lengths of the triangle.

congruence of triangles: Two triangles are congruent if their edge lengths respectively coincide.

dense: A subset A of a metric space (X, d) is called dense in X if every point of $X \setminus A$ is a limit point of A.

diameter: In a metric space (X, d) the diameter of a set $A \subset X$ is defined to be $\sup\{d(x, y) : x, y \in A\}$.

group: A set in which an associative binary operation is defined so that an identity and inverses to any element exist is called a group.

homeomorphic: Two topological spaces X and Y are called homeomorphic, if there is a bijection $f : X \to Y$ such that both f and f^{-1} are continuous.

identity: An identity (in a set A with a binary operation $\star$) is an element $e \in A$ such that $e \star a = a \star e = a$ for all $a \in A$.

interior point: A point z is called an interior point of a set $S \subset \mathbb{C}$ if there is an open disk about z which is contained in S.

inverse: The inverse of an element a (in a set A with a binary operation $\star$ and identity e) is an element $b \in A$ such that $a \star b = b \star a = e$.

isolated point: A point x in a metric space (X, d) is called an isolated point of a set $S \subset X$ if there exists an open disk B about x such that $B \cap S = \{x\}$.

limit point: A point x in a metric space (X, d) is called a limit point of a set $S \subset X$, if $B(x, 1/n) \cap S \setminus \{x\} \neq \emptyset$ for all $n \in \mathbb{N}$.

linear: A map F (defined on a vector space) is called linear if $F(\alpha u + \beta v) = \alpha F(u) + \beta F(v)$ for all $\alpha, \beta \in \mathbb{C}$ and all u, v in the domain of F.

local maximum: The value $u(z_0)$ is called a local maximum of a function $u : S \to \mathbb{R}$ if $u(z_0) \geq u(z)$ for all z in some neighborhood $V = B(z_0, r) \cap S$ of z_0.

monic: A polynomial is called monic if its leading coefficient is equal to 1.

monomial: A function of the form $m : \mathbb{C}^k \to \mathbb{C} : (z_1, ..., z_k) \mapsto a z_1^{n_1} ... z_k^{n_k}$ when a is a complex number and n_1, ..., n_k are non-negative integers. The number a is called the coefficient of the monomial while z_1, ..., z_k are called the variables.

periodic: A function f defined on $\mathbb{C}$ (or $\mathbb{R}$) is called periodic if there exists a complex (or real) number $a \neq 0$ such that $f(z+a) = f(z)$ for all $z \in \mathbb{C}$ (or $z \in \mathbb{R}$). The number a is called a period of f.

permutation: Let A be a finite set. A permutation of A is a bijective map from A to itself.

polynomial: A finite sum of monomials.

punctured disk: A set is called a punctured disk if it is a disk with the center removed.

rearrangement: The series $\sum_{n=1}^{\infty} w_n$ is a rearrangement of the series $\sum_{n=1}^{\infty} z_n$ if there is bijective sequence $k : \mathbb{N} \to \mathbb{N}$ such that $w_n = z_{k_n}$.

similarity of triangles: Two triangles are similar if the ratios of their respective edge lengths coincide.

singleton: A singleton is a set containing precisely one element.

symmetric polynomial: A polynomial is called symmetric if no permutation of its variables changes its values.

unit circle: The set of points in $\mathbb{C}$ of modulus 1.

unit disk: The set of points in $\mathbb{C}$ of modulus less than 1.

zero: x is called a zero of a function f if $f(x) = 0$.

List of special symbols

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