EGR 265-6D, Math Tools for Engineering Problem Solving
December 10, 2010, 1:30pm to 4:00pm

Name (Print last name first): .....

Student ID Number: ....

Final Exam

Problem 1 (8 points)

Find an explicit solution of the initial value problem

$$xyy' = 1 + y^2, \quad y(1) = 1.$$

Problem 2 (8 points)

A radioactive substance has a half-life of 10 days. The initial amount of the substance is 100 milligrams.
(a) Determine the decay rate of the substance.
(b) How much of the substance is left after 5 days?
(c) How long does it take for the substance to decay to 10 percent of its original amount?
Note: Write your answers in terms of natural logarithms, which do not need to be evaluated.

Problem 3 (14 points)

Consider the second order differential equation

$$y'' - 3y' + 2y = 10\cos(2x). \tag{1}$$

(a) Find the general solution of the homogeneous equation corresponding to (1).
(b) Find a particular solution of the inhomogeneous equation (1).
(c) Solve the initial value problem given by (1) and initial conditions $y(0) = 0$, $y'(0) = 0$.

Problem 4 (12 points)

A 100 cm spring measures 140 cm long after a mass of 10 kg is attached to it. The medium through which the mass moves offers a damping force with damping coefficient $\beta = 100$ kg/s. Include the correct units in all your answers below.
(a) Find the spring constant k, assuming that $g = 10$ m/s^2.
(b) Find the equation of motion of the mass if it is released from a position 10 cm below the equilibrium position with an upward velocity of 50 cm/s (choose the positive x-axis to be oriented downward).
(c) Will the mass return to the equilibrium position? If yes, when is the first time? If no, why not?

Problem 5 (10 points)

(a) Find the gradient of $f(x, y) = \sqrt{2x^2 + 2xy + y^2}$.
(b) Evaluate the directional derivative of $f(x, y)$ at the point $P(1, 2)$ in the direction from $P(1, 2)$ to $Q(2, 3)$.
(c) Find a unit vector in the direction of steepest decrease of $f(x, y)$ at the point $P(1, 2)$. Also find the rate of decrease in this direction.

Problem 6 (8 points)

Determine the equation of the tangent plane to the level surface $xyz = 6$ through the point $(1, 2, 3)$.

Problem 7 (8 points)

Find the work done by the force field

$$\mathbf{F}(x, y) = y^2\mathbf{i} + \frac{x^2}{y}\mathbf{j}$$

along the curve given by the graph of $y = e^x$, $0 \leq x \leq 1$.

Problem 8 (12 points)

(a) Verify that the force field $F(x, y) = (\cos x - \cos y)\mathbf{i} + x \sin y\mathbf{j}$ is conservative.
(b) Find a potential function $\phi(x, y)$ for $F(x, y)$.
(b) Find the work done by the force field $F(x, y)$ along the curve $x = t/2$, $y = (\pi - t)/2$, $0 \leq t \leq \pi$.

Problem 9 (12 points)

A lamina of constant density $\rho(x, y) = 1$ is bounded by the triangle with vertices $(0, 0)$, $(4, 0)$ and $(4, 2)$.
(a) Find the lamina's moment of inertia I_y with respect to the y-axis.
(b) Find the lamina's moment of inertia I_x with respect to the x-axis.

Problem 10 (8 points)

Let R be the region in the first quadrant which lies between the circles of radius $r = \sqrt{3}$ and $r = \sqrt{8}$. Find

$$\iint_R \sqrt{1 + x^2 + y^2}\, dA.$$