MA 227 (Calculus-III) Midterm test #2

Show your work. Each problem is 20 points Thu, Oct 14, 2004

1. Determine the largest set on which the function

$$f(x,y) = \begin{cases} \frac{x^2y^2}{x^4+y^4} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$$

is continuous. In particular, determine if $f(x,y)$ is continuous at the origin.

Answer: it is continuous everywhere except $(0,0)$. It is discontinuous at the origin because $\lim_{(x,y)\to(0,0)} f(x,y)$ does not exist (the limit value along the lines $x = 0$ and $y = 0$ equals zero, and the limit value along the line $x = y$ equals $1/2$).

2. Use Lagrange multipliers to find the maximum and the minimum value of the function $w = x^2 + y^2$ subject to the constraint $x^2 - 2x + y^2 - 4y = 0$.

For an extra credit, draw the curve $x^2 - 2x + y^2 - 4y = 0$ and indicate the points where the function w takes its minimum and maximum. Interpret the result in geometric terms.

Answer: minimum $w = 0$ at $(0, 0)$ and maximum $w = 20$ at $(2, 4)$.

For extra credit: the constraint defines a circle $(x - 1)^2 + (y - 2)^2 = 5$ centered at $(1, 2)$ and with radius $\sqrt{5}$. It passes through the origin, and the point $(2, 4)$ on the circle is the farthest from the origin.

3. Use the Chain Rule to find the partial derivatives $\partial z/\partial s$ and $\partial z/\partial t$:

$$z = y^3 \sin x, \qquad x = s^2 - st, \qquad y = \ln(2s - t)$$

Compute the values of these derivatives at the point $s = 1$, $t = 1$.

Answer:

$$\frac{\partial z}{\partial s} = (2s - t)[\ln(2s - t)]^3 \cos(s^2 - st) + 3[\ln(2s - t)]^2 \sin(s^2 - st)\, \frac{2}{2s - t}$$

and

$$\frac{\partial z}{\partial t} = -s\,[\ln(2s - t)]^3 \cos(s^2 - st) - 3[\ln(2s - t)]^2 \sin(s^2 - st)\, \frac{1}{2s - t}$$

At the point $s = t = 1$ both derivatives vanish (equal zero).

4. Find all critical points and use the second derivative test to determine local minima, local maxima, and saddle points of the function

$$f(x, y) = x^3 + y^3 + 3x^2 - 3y^2 - 8$$

Answer: a local maximum at $(-2, 0)$, a local minimum at $(0, 2)$, and saddle points at $(-2, 2)$ and $(0, 0)$.

5. Find the differential of the function

$$f(x,y) = \sqrt{x^4 - 12y^2}$$

Find the linearization $L(x,y)$ of f at the point $P = (2,1)$. Find the directional derivative of f at the point P in the direction of the vector $\mathbf{v} = (3,-4)$. Find the maximum rate of change of f at the point P and the direction in which it occurs.

Answers: the differential is

$$df = \frac{2x^3\,dx}{\sqrt{x^4 - 12y^2}} - \frac{12y\,dy}{\sqrt{x^4 - 12y^2}}$$

the linearization is

$$L(x,y) = 2 + 8(x-2) - 6(y-1)$$

The directional derivative is

$$D_{\mathbf{u}}f = \langle 8, -6\rangle \cdot \langle 3/5, -4/5\rangle = 48/5$$

The maximum rate of change is $\|\nabla f\| = 10$ in the direction of $\nabla f = \langle 8, -6\rangle$.